

A KLR GRADING OF THE BRAUER ALGEBRAS

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ABSTRACT. We construct a naturally \mathbb{Z} -graded algebra $\mathcal{G}_n(\delta)$ over R with KLR-like relations and give an explicit isomorphism between $\mathcal{G}_n(\delta)$ and $\mathcal{B}_n(\delta)$, the Brauer algebras over R , when R is a field of characteristic 0. This isomorphism allows us to exhibit a non-trivial \mathbb{Z} -grading on the Brauer algebras over a field of characteristic 0. As a byproduct of the proof, we also construct an explicit homogeneous cellular basis for $\mathcal{G}_n(\delta)$.

1. Introduction

Richard Brauer [2] introduced a class of finite dimensional algebras $\mathcal{B}_n(\delta)$ over a field R , which are called Brauer algebras, in order to study the n -th tensor power of the defining representations of orthogonal groups and symplectic groups. It is well known that the symmetric group algebras $R\mathfrak{S}_n$ is a subalgebra of $\mathcal{B}_n(\delta)$.

Khovanov and Lauda [11, 10] and Rouquier [16] have introduced a remarkable new family of algebras \mathcal{R}_n , the quiver Hecke algebras, for each oriented quiver, and they showed that they categorify the positive part of the enveloping algebras of the corresponding quantum groups. The algebras \mathcal{R}_n are naturally \mathbb{Z} -graded. Brundan and Kleshchev [3] proved that every degenerate and non-degenerate cyclotomic Hecke algebra H_n^Λ of type $G(r, 1, n)$ over a field is isomorphic to a cyclotomic quiver Hecke algebra \mathcal{R}_n^Λ of type A by constructing an explicit isomorphisms between these two algebras. Hu and Mathas [9] gave another proof of Brundan and Kleshchev's result using seminormal forms. Moreover, Hu and Mathas [8] defined a homogeneous basis of the cyclotomic quiver algebras \mathcal{R}_n^Λ which showed that H_n^Λ is a graded cellular algebra.

Because $R\mathfrak{S}_n$ is a special case of H_n^Λ , all above results hold in $R\mathfrak{S}_n$. It is natural to ask the question that whether Brauer algebras $\mathcal{B}_n(\delta)$ are graded cellular algebras. Ehrig and Stroppel [4] proved this result, but they were unable to give a presentation of the graded Brauer algebras similar to the KLR presentation of cyclotomic quiver Hecke algebras.

Let R be a field of characteristic 0. The main purpose of this paper is to construct a \mathbb{Z} -graded algebra over R with a parameter $\delta \in R$ analogues to the cyclotomic quiver Hecke algebras and to prove that this algebra is isomorphic to $\mathcal{B}_n(\delta)$. In Section 3 we define a \mathbb{Z} -graded algebra, $\mathcal{G}_n(\delta)$, by generators and relations. It is generated by elements

$$G_n(\delta) = \{e(\mathbf{i}) \mid \mathbf{i} \in P^n\} \cup \{\gamma_k \mid 1 \leq k \leq n\} \cup \{\psi_k \mid 1 \leq k \leq n-1\} \cup \{\epsilon_k \mid 1 \leq k \leq n-1\}, \quad (1.1)$$

and the relations are similar to the KLR relations for the cyclotomic quiver Hecke algebras of type A . In Section 4 we construct a set of homogeneous elements in $\mathcal{G}_n(\delta)$

$$\{\psi_{st} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}, \quad (1.2)$$

with $\deg \psi_{st} = \deg \mathbf{s} + \deg \mathbf{t}$, which are the Brauer-algebra-analogue of the graded cellular basis of the cyclotomic quiver Hecke algebras [8]. In Section 5 we prove:

Theorem A. *The algebra $\mathcal{G}_n(\delta)$ is spanned by the elements $\{\psi_{st} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$.*

We prove Theorem A via showing that

- (1) 1_R is a linear combination of (1.2) (cf. Proposition 4.39).
- (2) For any $(\lambda, f) \in \widehat{B}_n$, $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ and $a \in \mathcal{G}_n(\delta)$, we have

$$\psi_{st}a = \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{sv} + \sum_{\substack{(\mu, f) > (\lambda, f) \\ \mathbf{u}, \mathbf{v} \in \mathcal{T}_n^{ud}(\mu)}} c_{uv} \psi_{uv}, \quad (1.3)$$

where $c_v, c_{uv} \in R$ and $>$ is a total ordering on \widehat{B}_n (cf. Proposition 5.27).

Theorem A shows that $\dim \mathcal{G}_n(\delta) \leq (2n-1)!! = \dim \mathcal{B}_n(\delta)$ and as a byproduct, (1.3) shows that (1.2) has a cellular-like property.

In Section 6 we construct a new set of generators of the Brauer algebra $\mathcal{B}_n(\delta)$

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^n\} \cup \{\gamma_k \mid 1 \leq k \leq n\} \cup \{\psi_k \mid 1 \leq k \leq n-1\} \cup \{\epsilon_k \mid 1 \leq k \leq n-1\}, \quad (1.4)$$

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and in Section 7 we prove that the map $\mathcal{G}_n(\delta) \longrightarrow \mathcal{B}_n(\delta)$ given by sending the generators in (1.1) to those in (1.4) is a surjective algebra homomorphism. To show that the generators in (1.4) satisfy the relations of $\mathcal{G}_n(\delta)$ we make extensive use of seminormal forms of $\mathcal{B}_n(\delta)$, following the Hu-Mathas [9] approach in type A. In turn this relies Naz [14], or Rui-Si [17]. By construction our map $\mathcal{G}_n(\delta) \longrightarrow \mathcal{B}_n(\delta)$ is surjective and it is injective by Theorem A, we obtain the main result of this paper:

Theorem B. *Suppose R is a field of characteristic $p = 0$ and $\delta \in R$. Then $\mathcal{B}_n(\delta) \cong \mathcal{G}_n(\delta)$.*

By given $\mathcal{G}_n(\delta) \cong \mathcal{B}_n(\delta)$, (1.2) is a basis of $\mathcal{G}_n(\delta)$ because it spans $\mathcal{G}_n(\delta)$ by Theorem A and it has the right number of elements $(= (2n-1)!!)$. Because we proved the cellularity of (1.2) by (1.3), it is a graded cellular basis of $\mathcal{G}_n(\delta)$:

Theorem C. *The algebra $\mathcal{G}_n(\delta)$ is a graded cellular algebra with a graded cellular basis (1.2).*

Because $\mathcal{G}_n(\delta) \cong \mathcal{B}_n(\delta)$ and $\mathcal{B}_n^\Lambda \cong R\mathfrak{S}_n$, similar to the Brauer algebras, by removing the elements ϵ_k for $1 \leq k \leq n$, the quotient of $\mathcal{G}_n(\delta)$ is isomorphic to \mathcal{B}_n^Λ with weight Λ_k for any $k \in \mathbb{Z}$.

Finally we remark that the strategy that we use in this paper can be extended to the Brauer algebra over fields of positive characteristic, the degenerate cyclotomic Nazarov-Wenzl algebras and partition algebras. As this paper is already long enough, the details will appear in subsequent papers.

2. Preliminaries

2.1. The symmetric groups and Brauer algebras

Let \mathfrak{S}_n be the symmetric group acting on the integers $\{1, 2, \dots, n\}$. For $1 \leq k \leq n-1$, define $s_k = (k, k+1)$ as the elementary transpositions in \mathfrak{S}_n . Hence \mathfrak{S}_n is generated by

$$\{s_k \mid 1 \leq k \leq n-1\}$$

subject to the relations:

$$\begin{aligned} s_k^2 &= 1, & \text{for } 1 \leq k \leq n-1, \\ s_k s_r &= s_r s_k, & \text{for } 1 \leq k, r \leq n-1 \text{ and } |k-r| > 1, \\ s_k s_{k+1} s_k &= s_{k+1} s_k s_{k+1}, & \text{for } 1 \leq k \leq n-2. \end{aligned}$$

An expression $w = s_{i_1} s_{i_2} \dots s_{i_m}$ for w in terms of elementary transpositions is *reduced* if w cannot be expressed as a proper sub-expression of $s_{i_1} s_{i_2} \dots s_{i_m}$. For example, $w = s_2 s_3 s_2$ is a reduced expression and $w = s_2 s_3 s_2 s_3$ is not a reduced expression, because $w = s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_3 = s_3 s_2$. Notice that generally there are more than one reduced expressions for an element of \mathfrak{S}_n . For example, $s_2 s_3 s_2 = s_3 s_2 s_3$ and both of expressions are reduced.

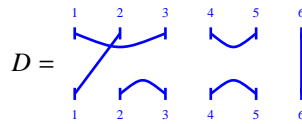
Let R be a commutative ring with identity 1 and $\delta \in R$. The Brauer algebra $\mathcal{B}_n(\delta)$ is a unital associative R -algebra with generators

$$\{s_1, s_2, \dots, s_{n-1}\} \cup \{e_1, e_2, \dots, e_{n-1}\},$$

associated with relations

- (1) (Inverses) $s_k^2 = 1$.
- (2) (Essential idempotent relation) $e_k^2 = \delta e_k$.
- (3) (Braid relations) $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ and $s_k s_r = s_r s_k$ if $|k-r| > 1$.
- (4) (Commutation relations) $s_k e_l = e_l s_k$ and $e_k e_r = e_r e_k$ if $|k-r| > 1$.
- (5) (Tangle relations) $e_k e_{k+1} e_k = e_k$, $e_{k+1} e_k e_{k+1} = e_{k+1}$, $s_k e_{k+1} e_k = s_{k+1} e_k$ and $e_k e_{k+1} s_k = e_k s_{k+1}$.
- (6) (Untwisting relations) $s_k e_k = e_k s_k = e_k$.

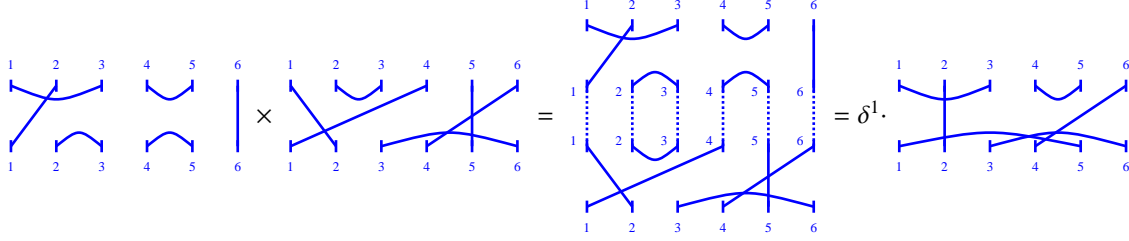
The symmetric group algebra $R\mathfrak{S}_n$ can be considered as a subalgebra of the Brauer algebra $\mathcal{B}_n(\delta)$ for any δ . The Brauer algebra $\mathcal{B}_n(\delta)$ (cf. [2], [18]) has R -basis consisting of Brauer diagrams D , which consist of two rows of n dots, labelled by $\{1, 2, \dots, n\}$, with each dot joined to one other dot. See the following diagram as an example:



Two diagrams D_1 and D_2 can be composed to get $D_1 \circ D_2$ by placing D_1 above D_2 and joining corresponding points and deleting all the interior loops. The multiplication of $\mathcal{B}_n(\delta)$ is defined by

$$D_1 \cdot D_2 = \delta^{n(D_1, D_2)} D_1 \circ D_2,$$

where $n(D_1, D_2)$ is the number of deleted loops. For example:



It is easy to see that we have $2n - 1$ possibilities to join the first dot with another one, then $2n - 3$ possibilities for the next dot and so on. So there are $(2n - 1)!! = (2n - 1) \cdot (2n - 3) \cdot \dots \cdot 3 \cdot 1$ number of Brauer diagrams, which implies the dimension of $\mathcal{B}_n(\delta)$ is $(2n - 1)!!$.

2.2. (Graded) cellular algebras

Following [7], we now introduce the graded cellular algebras. Reader may also refer to [8]. Let R be a commutative ring with 1 and let A be a unital R -algebra.

2.1. Definition. A *graded cell datum* for A is a triple (Λ, T, C, \deg) where $\Lambda = (\Lambda, >)$ is a poset, either finite or infinite, and $T(\lambda)$ is a finite set for each $\lambda \in \Lambda$, \deg is a function from $\coprod_{\lambda} T(\lambda)$ to \mathbb{Z} , and

$$C: \prod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \longrightarrow A$$

is an injective map which sends (s, t) to a_{st}^λ such that:

- (1) $\{a_{st}^\lambda \mid \lambda \in \Lambda, s, t \in T(\lambda)\}$ is an R -free basis of A ;
- (2) for any $r \in A$ and $t \in T(\lambda)$, there exists scalars $c_t^\vee(r)$ such that, for any $s \in T(\lambda)$,

$$a_{st}^\lambda \cdot r \equiv \sum_{v \in T(\lambda)} c_t^\vee(r) a_{sv}^\lambda \pmod{A^{>\lambda}}$$

where $A^{>\lambda}$ is the R -submodule of A spanned by $\{a_{xy}^\mu \mid \mu > \lambda, x, y \in T(\mu)\}$;

- (3) the R -linear map $*$: $A \longrightarrow A$ which sends a_{st}^λ to a_{ts}^λ , for all $\lambda \in \Lambda$ and $s, t \in T(\lambda)$, is an anti-isomorphism of A .
- (4) each basis element a_{st}^λ is homogeneous of degree $\deg a_{st}^\lambda = \deg s + \deg t$, for $\lambda \in \Lambda$ and all $s, t \in T(\lambda)$.

If a graded cell datum exists for A then A is a *graded cellular algebra*. Similarly, by forgetting the grading we can define a *cell datum* and hence a *cellular algebra*.

Suppose A is a graded cellular algebra with graded cell datum (Λ, T, C, \deg) . For any $\lambda \in \Lambda$, define $A^{\geq \lambda}$ to be the R -submodule of A spanned by

$$\{c_{st}^\mu \mid \mu \geq \lambda, s, t \in T(\mu)\}.$$

Then $A^{>\lambda}$ is an ideal of $A^{\geq \lambda}$ and hence $A^{\geq \lambda}/A^{>\lambda}$ is a A -module. For any $s \in T(\lambda)$ we define C_s^λ to be the A -submodule of $A^{\geq \lambda}/A^{>\lambda}$ with basis $\{a_{st}^\lambda + A^{>\lambda} \mid t \in T(\lambda)\}$. By the cellularity of A we have $C_s^\lambda \cong C_t^\lambda$ for any $s, t \in T(\lambda)$.

2.2. Definition. Suppose $\lambda \in \mathcal{P}_n^\Lambda$. Define the *cell module* of A to be $C^\lambda = C_s^\lambda$ for any $s \in T(\lambda)$, which has basis $\{a_t^\lambda \mid t \in T(\lambda)\}$ and for any $r \in A$,

$$a_t^\lambda \cdot r = \sum_{u \in T(\lambda)} c_u^r a_u^\lambda$$

where c_u^r are determined by

$$a_{st}^\lambda \cdot r = \sum_{u \in T(\lambda)} c_u^r a_{su}^\lambda + A^{>\lambda}.$$

We can define a bilinear map $\langle \cdot, \cdot \rangle: C^\lambda \times C^\lambda \longrightarrow \mathbb{Z}$ such that

$$\langle a_s^\lambda, a_t^\lambda \rangle a_{uv}^\lambda = a_{us}^\lambda a_{tv}^\lambda + A^{>\lambda}$$

and let $\text{rad } C^\lambda = \{s \in C^\lambda \mid \langle s, t \rangle = 0 \text{ for all } t \in C^\lambda\}$. As $\langle \cdot, \cdot \rangle$ is homogeneous of degree 0, $\text{rad } C^\lambda$ is a graded A -submodule of C^λ .

2.3. Definition. Suppose $\lambda \in \mathcal{P}_n^\Lambda$. Let $D^\lambda = C^\lambda / \text{rad } C^\lambda$ as a graded A -module.

Exactly as in the ungraded case [7, Theorem 3.4], we obtain the following:

2.4. Theorem (Hu-Mathas [8, Theorem 2.10]). *The set $\{D^\lambda \langle k \rangle \mid \lambda \in \Lambda, D^\lambda \neq 0, k \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple A -modules.*

In particular, the symmetric group algebra $R\mathfrak{S}_n$ is a graded cellular algebra. The details of the graded cellular structure of $R\mathfrak{S}_n$ will be introduced in Section 2.5. It is well-known that the Brauer algebras $\mathcal{B}_n(\delta)$ are (ungraded) cellular algebras [7, Theorem 4.10]. In the following two subsections we will construct the cellular basis of the Brauer algebras.

2.3. Combinatorics

In this subsection we introduce the combinatorics of up-down tableaux, which will be used to index the cellular basis of the Brauer algebras. Throughout the rest of this paper, we fix R to be a field with characteristic 0 and $\delta \in R$.

Recall that a *partition* of n is a weakly decreased sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \dots = n$. In such case we denote $\lambda \vdash n$. Because $|\lambda| < \infty$, there are finite many nonzero λ_i for $i \geq 1$. Because $\lambda_1 \geq \lambda_2 \geq \dots$, there exists $k \geq 1$ such that $\lambda_k \geq 0$ and $\lambda_{k+1} = \lambda_{k+2} = \dots = 0$. Usually we will write $\lambda = (\lambda_1, \dots, \lambda_k)$ instead of an infinite sequence.

Let \widehat{H}_n be the set of all partitions of n . We can define a partial ordering \trianglelefteq on \widehat{H}_n , which is called the *dominance ordering*. Given $\lambda, \mu \in \widehat{H}_n$, we say $\lambda \trianglelefteq \mu$ if for any $k \geq 1$, we have $|\sum_{i=1}^k \lambda_i| \leq |\sum_{i=1}^k \mu_i|$. Write $\lambda \triangleleft \mu$ if $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$. The dominance ordering can be extended to a total ordering \leq , the *lexicographic ordering*. We write $\lambda < \mu$ if there exists k such that $\lambda_i = \mu_i$ for all $i < k$ and $\lambda_k < \mu_k$. Define $\lambda \leq \mu$ if $\lambda < \mu$ or $\lambda = \mu$. Then $\lambda \trianglelefteq \mu$ implies $\lambda \leq \mu$.

The *Young diagram* of a partition λ is the set

$$[\lambda] := \{(r, l) \mid 1 \leq l \leq \lambda_r\}.$$

For example, $\lambda = (3, 2, 2)$ is a partition of 7, and the Young diagram of λ is

$$[\lambda] = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & \square & \\ \hline \end{array}.$$

Define $\alpha = \pm(r, l)$ to be a *node* for positive integers r and l , and denote $\alpha > 0$ if $\alpha = (r, l)$ and $\alpha < 0$ if $\alpha = -(r, l)$. In this paper we allow to work with linear combination of nodes. In more details, suppose α and β are two nodes, we write $\alpha + \beta = 0$ if $\alpha = (r, l)$ and $\beta = -(r, l)$, or vise versa. Similarly, we write $\alpha = -\beta$ if $\alpha + \beta = 0$.

Suppose $\alpha = (r, l)$. We say α is a node of λ in row r and column l if $\alpha \in [\lambda]$. For example, let $\alpha = (2, 2)$, $\beta = (2, 3)$ and $\lambda = (3, 2, 2)$. Then α is a node of λ in row 2 and column 2, and β is not a node of λ .

Suppose λ is a partition. A node $\alpha > 0$ is *addable* if $\lambda \cup \{\alpha\}$ is still a partition, and it is *removable* if $\lambda \setminus \{\alpha\}$ is still a partition. Let $\mathcal{A}(\lambda)$ and $\mathcal{R}(\lambda)$ be the sets of addable and removable nodes of λ , respectively, and set $\mathcal{AR}(\lambda) = \mathcal{A}(\lambda) \cup \mathcal{R}(\lambda)$.

A λ -*tableau* is any bijection $t: \{1, 2, \dots, n\} \rightarrow [\lambda]$. We identify a λ -tableau t with a labeling of the diagram of λ . That is, we label the node $(r, l) \in [\lambda]$ with the integer $t^{-1}(r, l)$. We say a tableau t has shape λ if it is a λ -tableau. A tableau t is *standard* if the entries of each row and each column of t increase. Suppose $\lambda \vdash n$. Denote $\text{Std}(\lambda)$ to be the set of all standard tableau of shape λ .

Define $\widehat{B}_n := \{(\lambda, f) \mid \lambda \in \widehat{H}_{n-2f} \text{ and } 0 \leq f \leq \lfloor \frac{n}{2} \rfloor\}$ and \widehat{B} to be the graph with

- (1) vertices at level n : \widehat{B}_n , and
- (2) an edge $(\lambda, f) \rightarrow (\mu, m)$, $(\lambda, f) \in \widehat{B}_{n-1}$ and $(\mu, m) \in \widehat{B}_n$, if either μ is obtained by adding a node to λ , or by deleting a node from λ .

We can extend the dominance ordering and lexicographic ordering of partitions to \widehat{B}_n by defining $(\lambda, f) \trianglelefteq (\mu, m)$ if $f < m$, or $f = m$ and $\lambda \trianglelefteq \mu$; and $(\lambda, f) \leq (\mu, m)$ if $f < m$, or $f = m$ and $\lambda \leq \mu$. We define \triangleleft and $<$ similarly.

2.5. Definition. Let $(\lambda, f) \in \widehat{B}_n$. An *up-down tableau* of shape (λ, f) is a sequence

$$t = ((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \dots, (\lambda^{(n)}, f_n)), \quad (2.1)$$

where $(\lambda^{(0)}, f_0) = (\emptyset, 0)$, $(\lambda^{(n)}, f_n) = (\lambda, f)$ and $(\lambda^{(k-1)}, f_{k-1}) \rightarrow (\lambda^{(k)}, f_k)$ is an edge in \widehat{B} , for $k = 1, \dots, n$. We write $\text{Shape}(t) = (\lambda, f)$. If $k = 0, 1, \dots, n$, we denote $t_k = \lambda^{(k)}$ and define the truncation of t to level k to be the up-down tableau

$$t|_k = ((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \dots, (\lambda^{(k)}, f_k)).$$

For any $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$ and $\lambda \vdash n - 2f$, define

$$\mathcal{T}_n^{ud}(\lambda) := \{t \mid t \text{ is an up-down tableau of shape } (\lambda, f) \in \widehat{B}_n\}.$$

Suppose $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. We define the dominance ordering $\mathbf{s} \leq \mathbf{t}$ if $s_k \leq t_k$ for any k with $1 \leq k \leq n$ and $\mathbf{s} < \mathbf{t}$ if $\mathbf{s} \leq \mathbf{t}$ and $\mathbf{s} \neq \mathbf{t}$.

An up-down tableau $\mathbf{t} = ((\lambda^{(0)}, f_0), (\lambda^{(1)}, f_1), \dots, (\lambda^{(n)}, f_n))$ can be identified with a n -tuple of nodes:

$$\mathbf{t} = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad (2.2)$$

where $\alpha_k = (r, l)$ if $\lambda^{(k)} = \lambda^{(k-1)} \cup \{(r, l)\}$ and $\alpha_k = -(r, l)$ if $\lambda^{(k)} = \lambda^{(k-1)} \setminus \{(r, l)\}$ for $1 \leq k \leq n$. Therefore an up-down tableau \mathbf{t} can be identified as a map $\mathbf{t}: \{1, 2, \dots, n\} \rightarrow \{\pm(r, l) \mid r, l \in \mathbb{Z}_{>0}\}$. Note that the range of \mathbf{t} is all (positive and negative) nodes and \mathbf{t} is not necessary injective. We have $\mathbf{t}(k) = \alpha_k$ for $1 \leq k \leq n$.

We define a right action of \mathfrak{S}_n on the up-down tableaux of n . Suppose $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$ and $1 \leq k \leq n-1$. Define $\mathbf{t} \cdot s_k = (\alpha_1, \dots, \alpha_{k-2}, \alpha_k, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_n)$. We note that $\mathbf{t} \cdot s_k$ is not necessarily an up-down tableau, and when $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$, then $\mathbf{t} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$ if $\mathbf{t} \cdot s_k$ is an up-down tableau.

Two nodes $\alpha = (i, j) > 0$ and $\beta = (r, l) > 0$ are *adjacent* if $i = r \pm 1$ and $j = l$, or $i = r$ and $j = l \pm 1$. The next Lemma can be verified directly by the construction of up-down tableaux. It gives conditions for $\mathbf{t} \cdot s_k$ to be an up-down tableau.

2.6. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. For $1 \leq k \leq n-1$, $\mathbf{t} \cdot s_k$ is an up-down tableau if and only if one of the following conditions hold:

- (1) $\mathbf{t}(k) > 0$, $\mathbf{t}(k+1) > 0$, and $\mathbf{t}(k)$ and $\mathbf{t}(k+1)$ are not adjacent.
- (2) $\mathbf{t}(k) < 0$, $\mathbf{t}(k+1) < 0$, and $-\mathbf{t}(k)$ and $-\mathbf{t}(k+1)$ are not adjacent.
- (3) $\mathbf{t}(k) > 0$, $\mathbf{t}(k+1) < 0$, and $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$.
- (4) $\mathbf{t}(k) < 0$, $\mathbf{t}(k+1) > 0$, and $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$.

Recall $\delta \in R$ and let x be an indeterminate. Suppose $\alpha = (r, l)$ is a positive node. The *content* of α is $\text{cont}(\alpha) = \frac{x-1}{2} + l - r$ and the *residue* of α , $\text{res}(\alpha)$, is the evaluation of the content at $x = \delta$. Set $\text{cont}(-\alpha) = -\text{cont}(\alpha)$ and $\text{res}(-\alpha) = -\text{res}(\alpha)$.

Suppose $\mathbf{t} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is an up-down tableau. Define $c_t(k) = \text{cont}(\alpha_k)$ and $r_t(k) = \text{res}(\alpha_k)$ for $1 \leq k \leq n$. We define the *residue sequence* of \mathbf{t} to be $\mathbf{i}_t = (i_1, i_2, \dots, i_n)$ such that $i_k = r_t(k)$ for any $1 \leq k \leq n$. Let $P = \frac{\delta-1}{2} + \mathbb{Z}$. One can see that $\text{res}(\alpha) \in P$ for any node α . Therefore one can see that the residue sequence $\mathbf{i}_t \in P^n$. Suppose $\mathbf{i} \in P^n$. Let $\mathcal{T}_n^{ud}(\mathbf{i})$ be the set containing all the up-down tableaux with residue sequence \mathbf{i} .

Here we give the notations for subtraction and concatenation of n -tuples in P^n . For $\mathbf{i} = (i_1, \dots, i_n) \in P^n$, denote $\mathbf{i}|_k = (i_1, \dots, i_k) \in P^k$ for $1 \leq k \leq n$; and denote $\mathbf{i} \vee i = (i_1, \dots, i_n, i) \in P^{n+1}$ for $i \in P$.

We now introduce the degree function of the set of up-down tableaux. Ultimately this degree function will describe the grading on $\mathcal{B}_n(\delta)$.

Suppose we have $(\lambda, f) \rightarrow (\mu, m)$. Write $\lambda \ominus \mu = \alpha$ if $\lambda = \mu \cup \{\alpha\}$ or $\mu = \lambda \cup \{\alpha\}$. For any up-down tableau \mathbf{t} and an integer k , with $1 \leq k \leq n$, let $\lambda = \mathbf{t}_{k-1}$, $\mu = \mathbf{t}_k$ and $\alpha = (r, l) = \lambda \ominus \mu$. Define

$$\begin{aligned} \mathcal{A}_t(k) &= \{\beta = (k, c) \in \mathcal{A}(\lambda) \mid \text{res}(\beta) = \text{res}(\alpha) \text{ and } k > r\}, & \text{if } \mu = \lambda \cup \{\alpha\}, \\ \widehat{\mathcal{A}}_t(k) &= \{\beta = (k, c) \in \mathcal{A}(\mu) \mid \text{res}(\beta) = -\text{res}(\alpha) \text{ and } k \neq r\}, & \text{if } \mu = \lambda \setminus \{\alpha\}; \\ \mathcal{R}_t(k) &= \{\beta = (k, c) \in \mathcal{R}(\lambda) \mid \text{res}(\beta) = \text{res}(\alpha) \text{ and } k > r\}, & \text{if } \mu = \lambda \cup \{\alpha\}, \\ \widehat{\mathcal{R}}_t(k) &= \{\beta = (k, c) \in \mathcal{R}(\mu) \mid \text{res}(\beta) = -\text{res}(\alpha)\}, & \text{if } \mu = \lambda \setminus \{\alpha\}. \end{aligned}$$

2.7. Definition. Suppose \mathbf{t} is an up-down tableau of size n . For integer k with $0 \leq k \leq n-1$, write $\lambda = \mathbf{t}_{k-1}$, $\mu = \mathbf{t}_k$ and $\alpha = \lambda \ominus \mu$. Define

$$\deg(\mathbf{t}|_{k-1} \Rightarrow \mathbf{t}|_k) := \begin{cases} |\mathcal{A}_t(k)| - |\mathcal{R}_t(k)|, & \text{if } \mu = \lambda \cup \{\alpha\}, \\ |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}}, & \text{if } \mu = \lambda \setminus \{\alpha\}, \end{cases}$$

and the *degree* of \mathbf{t} is

$$\deg \mathbf{t} := \sum_{k=1}^n \deg(\mathbf{t}|_{k-1} \Rightarrow \mathbf{t}|_k).$$

2.8. Remark. We note that when the characteristic of the field is 0, we have $|\mathcal{A}_t(k)| = |\mathcal{R}_t(k)| = 0$ for any $1 \leq k \leq n$. Therefore, we always have $\deg(\mathbf{t}|_{k-1} \Rightarrow \mathbf{t}|_k) = 0$ when $\mu = \lambda \cup \{\alpha\}$.

2.9. Example. Let $n = 6$, $\lambda = (1, 1)$, $\delta = 1$ and $\mathbf{t} = (\emptyset, \square, \square\square, \square\square, \square\square, \square\square) \in \mathcal{T}_n^{ud}(\lambda)$. By Remark 2.8, we have $\deg \mathbf{t} = \sum_k \deg(\mathbf{t}|_{k-1} \Rightarrow \mathbf{t}|_k)$, where k take values such that $\mathbf{t}|_k$ is obtained by removing a node from $\mathbf{t}|_{k-1}$. Therefore, we have $\deg \mathbf{t} = \deg(\mathbf{t}|_4 \Rightarrow \mathbf{t}|_5) + \deg(\mathbf{t}|_5 \Rightarrow \mathbf{t}|_6)$.

By the definitions, we have $\widehat{\mathcal{A}}_t(5) = \widehat{\mathcal{R}}_t(5) = \emptyset$, $\widehat{\mathcal{A}}_t(6) = \emptyset$ and $\widehat{\mathcal{R}}_t(6) = \{(2, 1)\}$. Because $\delta = 1$, for any node α , we have $\text{res}(\alpha) \in \mathbb{Z}$, which implies $\delta_{\text{res}(\alpha), -\frac{1}{2}} = 0$. Hence, the degree of \mathbf{t} is

$$\deg \mathbf{t} = \deg(\mathbf{t}|_4 \Rightarrow \mathbf{t}|_5) + \deg(\mathbf{t}|_5 \Rightarrow \mathbf{t}|_6) = 0 - 1 = -1.$$

2.10. Example. Let $n = 6$, $\lambda = (1, 1)$, $\delta = 0$ and $\mathbf{t} = (\emptyset, \square, \square\square, \square\square, \square\square, \square\square) \in \mathcal{T}_n^{ud}(\lambda)$. Following the same argument as in Example 2.9, we have $\deg \mathbf{t} = \deg(\mathbf{t}|_4 \Rightarrow \mathbf{t}|_5) + \deg(\mathbf{t}|_5 \Rightarrow \mathbf{t}|_6)$.

By the definitions, we have $\mathcal{A}_t(5) = \emptyset$ and $\mathcal{B}_t(5) = \{(1, 2)\}$. If we set $\lambda = \mathbf{t}_4$ and $\mu = \mathbf{t}_5$, we have $\mu = \lambda \setminus \{\alpha\}$ where $\alpha = (2, 2)$. Because $\text{res}(\alpha) = -\frac{1}{2}$, we have

$$\deg(\mathbf{t}|_4 \Rightarrow \mathbf{t}|_5) = |\widehat{\mathcal{A}}_t(5)| - |\widehat{\mathcal{B}}_t(5)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = 0 - 1 + 1 = 0.$$

Similarly, we have $\widehat{\mathcal{A}}_t(6) = \widehat{\mathcal{B}}_t(6) = \emptyset$. If we set $\lambda = \mathbf{t}_5$ and $\mu = \mathbf{t}_6$, we have $\mu = \lambda \setminus \{\alpha\}$ where $\alpha = (1, 2)$. Because $\text{res}(\alpha) = \frac{1}{2}$, we have

$$\deg(\mathbf{t}|_5 \Rightarrow \mathbf{t}|_6) = |\widehat{\mathcal{A}}_t(6)| - |\widehat{\mathcal{B}}_t(6)| + \delta_{\text{res}(\alpha), \frac{1}{2}} = 0.$$

Hence, the degree of \mathbf{t} is $\deg \mathbf{t} = \deg(\mathbf{t}|_4 \Rightarrow \mathbf{t}|_5) + \deg(\mathbf{t}|_5 \Rightarrow \mathbf{t}|_6) = 0 + 0 = 0$.

2.4. Jucys-Murphy elements and Cellularity of Brauer algebras

In the Brauer algebra $\mathcal{B}_n(\delta)$, Nazarov [14] defined *Jucys-Murphy elements* L_k for $1 \leq k \leq n$ by $L_1 = \frac{\delta-1}{2}$ and

$$L_{k+1} = s_k - e_k + s_k L_k s_k, \quad \text{for } 1 \leq k \leq n-1.$$

2.11. Lemma (Nazarov [14]). *The following relations hold in the algebra $\mathcal{B}_n(\delta)$:*

$$\begin{aligned} s_k L_r &= L_r s_k, & e_k L_r &= L_r e_k; & r &\neq k, k+1; \\ s_k L_k - L_{k+1} s_k &= e_k - 1, & L_k s_k - s_k L_{k+1} &= e_k - 1; \\ e_k(L_k + L_{k+1}) &= 0, & (L_k + L_{k+1})e_k &= 0. \end{aligned}$$

Graham and Lehrer [7] proved that $\mathcal{B}_n(\delta)$ is a cellular algebra over any commutative ring R . Enyang [5, 6] constructed another cellular basis indexed by pairs (\mathbf{s}, \mathbf{t}) , where $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ and $(\lambda, f) \in \widehat{B}_n$. We will only state the Theorem here.

2.12. Theorem (Enyang [6]). *Let $\mathcal{B}_n(\delta)$ be a Brauer algebra over a commutative ring R and $*$: $\mathcal{B}_n(\delta) \rightarrow \mathcal{B}_n(\delta)$ be the R -linear involution which fixes s_k and e_k for $1 \leq k \leq n-1$. Then $\mathcal{B}_n(\delta)$ has a cellular basis*

$$\{ m_{\mathbf{st}} \mid \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda), (\lambda, f) \in \widehat{B}_n \}$$

such that

$$m_{\mathbf{st}} L_k = r_t(k) m_{\mathbf{st}} + \sum_{\substack{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda) \\ \mathbf{v} \triangleright \mathbf{t}}} c_{\mathbf{v}} m_{\mathbf{sv}} + \sum_{\substack{\mathbf{u}, \mathbf{v} \in \mathcal{T}_n^{ud}(\mu) \\ (\mu, m) \in \widehat{B}_n \\ (\mu, m) \triangleright (\lambda, f)}} c_{\mathbf{uv}} m_{\mathbf{uv}}.$$

2.5. Seminormal forms and idempotents

In this subsection we develop the theory of seminormal forms for Brauer algebras, summarizing results that are in the literature, such as [12, 17].

Recall that $\mathcal{B}_n(\delta)$ is a R -algebra, where R is a field of characteristic 0. Define $\mathbb{F} = R(x)$ to be the rational field with indeterminate x and $\mathcal{O} = R[x]_{(x-\delta)} = R[[x-\delta]]$. Let $\mathfrak{m} = (x-\delta)\mathcal{O} \subset \mathcal{O}$. Then \mathfrak{m} is a maximal ideal of \mathcal{O} and $R \cong \mathcal{O}/\mathfrak{m}$.

Let $\mathcal{B}_n^{\mathbb{F}}(x)$ and $\mathcal{B}_n^{\mathcal{O}}(x)$ be the Brauer algebras over \mathbb{F} and \mathcal{O} , respectively. Then $\mathcal{B}_n^{\mathbb{F}}(x) = \mathcal{B}_n^{\mathcal{O}}(x) \otimes_{\mathcal{O}} \mathbb{F}$ and $\mathcal{B}_n(\delta) \cong \mathcal{B}_n^{\mathcal{O}}(x) \otimes_R \mathbb{F} \cong \mathcal{B}_n^{\mathcal{O}}(x)/(x-\delta)\mathcal{B}_n^{\mathcal{O}}(x)$. In order to avoid confusion we will write the generators of $\mathcal{B}_n^{\mathcal{O}}(x)$ and $\mathcal{B}_n^{\mathbb{F}}(x)$ as $s_k^{\mathcal{O}}$ and $e_k^{\mathcal{O}}$ and generators of $\mathcal{B}_n(\delta)$ as s_k and e_k . Hence for any element $w \in \mathcal{B}_n(\delta)$, we write $w^{\mathcal{O}} = w \otimes_R 1_{\mathcal{O}} \in \mathcal{B}_n^{\mathcal{O}}(x)$, so that $w = w^{\mathcal{O}} \otimes_{\mathcal{O}} 1_R$.

Because $\mathcal{B}_n(\delta) \cong \mathcal{B}_n^{\mathcal{O}}(x) \otimes_{\mathcal{O}} \mathbb{F} \cong \mathcal{B}_n^{\mathcal{O}}(x)/(x-\delta)\mathcal{B}_n^{\mathcal{O}}(x)$, if $x, y \in \mathcal{B}_n^{\mathcal{O}}(x)$ and we have $x \equiv y \pmod{(x-\delta)\mathcal{B}_n^{\mathcal{O}}(x)}$, then $x \otimes_{\mathcal{O}} 1_R = y \otimes_{\mathcal{O}} 1_R$ as elements of $\mathcal{B}_n(\delta)$. This observation will give us a way to extend the results of $\mathcal{B}_n^{\mathcal{O}}(x)$ to $\mathcal{B}_n(\delta)$.

The next Lemma says that an up-down tableau \mathbf{t} is completely determined by its contents $c_t(k)$ for $1 \leq k \leq n$.

2.13. Lemma. *Suppose \mathbf{s}, \mathbf{t} are up-down tableaux of size n . Then $\mathbf{s} = \mathbf{t}$ if and only if $c_s(k) = c_t(k)$ for all $1 \leq k \leq n$.*

Hence $\mathcal{B}_n^{\mathbb{F}}(x)$ and \mathbb{F} satisfies the separation condition in the sense of Mathas [12, Definition 2.8] or Rui-Si [17, Assumption 3.1]. The results we have in the rest of this subsection are already included in Mathas [12, Section 3, 4] and Rui-Si [17, Section 3].

2.14. Definition. Suppose $\mathcal{B}_n^{\mathbb{F}}(x)$ is the Brauer algebra over \mathbb{F} and $\{ m_{\mathbf{st}} \mid \mathcal{T}_n^{ud}(\lambda), (\lambda, f) \in \widehat{B}_n \}$ is the basis of $\mathcal{B}_n^{\mathbb{F}}(x)$.

(1) For any $1 \leq k \leq n$, define $\mathcal{C}(k) = \{ c_t(k) \mid \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda), (\lambda, f) \in \widehat{B}_n \}$.

(2) $F_{\mathbf{t}} = \prod_{k=1}^n \prod_{\substack{c \in \mathcal{C}(k) \\ c_t(k) \neq c}} \frac{L_k^{\mathcal{O}} - c}{c_t(k) - c}.$

$$(3) f_{st} = F_s m_{st} F_t,$$

where $s, t \in \mathcal{T}_n^{ud}(\lambda)$ and $(\lambda, f) \in \widehat{B}_n$.

By Theorem 2.12, $f_{st} = m_{st} + \sum_{u \triangleright s, v \triangleright t} r_{uv} m_{uv}$, for some $r_{uv} \in \mathbb{F}$. Therefore

$$\{f_{st} \mid s, t \in \mathcal{T}_n^{ud}(\lambda), (\lambda, f) \in \widehat{B}_n\}$$

is a basis of $\mathcal{B}_n^{\mathbb{R}}(x)$. This basis is called the *seminormal basis* of $\mathcal{B}_n(\delta)$; see [12, Theorem 3.7].

2.15. Lemma. *Suppose that $\mathcal{B}_n^{\mathbb{R}}(x)$ is the Brauer algebra over \mathbb{F} . Then we have*

$$f_{st} L_k^{\mathcal{O}} = c_t(k) f_{st}, \quad L_k^{\mathcal{O}} f_{st} = c_s(k) f_{st}, \quad \text{and} \quad f_{st} f_{uv} = \delta_{t,u} f_{sv},$$

for $1 \leq k \leq n$ and $s, t, u, v \in \mathcal{T}_n^{ud}(\lambda)$.

Nazarov [14] gave the actions of $s_k^{\mathcal{O}}$ and $e_k^{\mathcal{O}}$ on the f_{st} 's. Readers may also check Rui-Si [17]. Suppose $s, t \in \mathcal{T}_n^{ud}(\lambda)$ and $(\lambda, f) \in \widehat{B}_n$. Define

$$f_{st} e_k^{\mathcal{O}} = \sum_{u \in \mathcal{T}_n^{ud}(\lambda)} e_k(t, u) f_{su} \quad \text{and} \quad f_{st} s_k^{\mathcal{O}} = \sum_{u \in \mathcal{T}_n^{ud}(\lambda)} s_k(t, u) f_{su}.$$

2.16. Definition. Suppose $1 \leq k \leq n-1$ and $(\lambda, f) \in \widehat{B}_n$. For $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t_{k-1} = t_{k+1}$, define an equivalence relation \sim^k by declaring that $t \sim^k s$ if $t_r = s_r$ whenever $1 \leq r \leq n$ and $r \neq k$, for $s \in \mathcal{T}_n^{ud}(\lambda)$.

Suppose $\mathbf{i}, \mathbf{j} \in P^n$ with $i_k + i_{k+1} = j_k + j_{k+1} = 0$ for some $1 \leq k \leq n-1$. We define an equivalence relation \sim^k on P^n by declaring that $\mathbf{i} \sim^k \mathbf{j}$ if $i_r = j_r$ whenever $1 \leq r \leq n$ and $r \neq k, k+1$. It is easy to see that $t \sim^k s$ only if $s_{k-1} = s_{k+1} = t_{k-1} = t_{k+1}$ and $\mathbf{i}_t \sim^k \mathbf{i}_s$. The next result is a special case of [1, 4.2].

2.17. Lemma. *Suppose $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t_{k-1} = t_{k+1} = \mu$. Then there is a bijection between $\mathcal{AR}(\mu)$ and the set $\{s \in \mathcal{T}_n^{ud}(\lambda) \mid s \sim^k t\}$.*

Suppose $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t_{k-1} = t_{k+1}$ for some k with $1 \leq k \leq n-1$. Define

$$e_k(t, t) := (2c_t(k) + 1) \prod_{\substack{u \stackrel{k}{\sim} t \\ u \neq t}} \frac{c_t(k) + c_u(k)}{c_t(k) - c_u(k)} \in \mathbb{F}.$$

2.18. Theorem. *Suppose $(\lambda, f) \in \widehat{B}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$. For any $1 \leq k \leq n-1$ and $s \in \mathcal{T}_n^{ud}(\lambda)$, we have:*

(1) *If $t_{k-1} \neq t_{k+1}$ and $t \cdot s_k$ does not exist, then*

$$f_{st} s_k^{\mathcal{O}} = \frac{1}{c_t(k+1) - c_t(k)} f_{st}.$$

(2) *If $t_{k-1} \neq t_{k+1}$ and $u = t \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$, then*

$$f_{st} s_k^{\mathcal{O}} = \begin{cases} \frac{1}{c_t(k+1) - c_t(k)} f_{st} + f_{su}, & \text{if } t \triangleright u, \\ \frac{1}{c_t(k+1) - c_t(k)} f_{st} + (1 - \frac{1}{(c_s(k+1) - c_s(k))^2}) f_{su}, & \text{if } u \triangleright t. \end{cases}$$

(3) *If $t_{k-1} \neq t_{k+1}$, then $f_{st} e_k^{\mathcal{O}} = 0$.*

(4) *If $t_{k-1} = t_{k+1}$, then*

$$f_{st} s_k^{\mathcal{O}} = \sum_{u \stackrel{k}{\sim} t} s_k(t, u) f_{su} = \sum_{u \stackrel{k}{\sim} t} \frac{e_k(t, u) - \delta_{tu}}{c_t(k) + c_u(k)} f_{su}.$$

(5) *If $t_{k-1} = t_{k+1}$, then*

$$f_{st} e_k^{\mathcal{O}} = \sum_{u \stackrel{k}{\sim} t} e_k(t, u) f_{su}.$$

(6) *If $t_{k-1} = t_{k+1}$ and $u \sim^k t \sim^k v$, then*

$$e_k(u, t) e_k(t, v) = e_k(u, v) e_k(t, t).$$

The following result gives an explicit construction on (central) primitive idempotents of $\mathcal{B}_n^{\mathbb{R}}(x)$. Such result has been proved by Mathas [12, Theorem 3.16] for general cellular algebras under separation condition.

For $t \in \mathcal{T}_n^{ud}(\lambda)$, define $\gamma_t \in \mathbb{F}$ such that $f_{tt} f_{tt} = \gamma_t f_{tt}$. By the cellularity of $\{f_{st}\}$, we have $f_{st} f_{tu} = \gamma_t f_{su}$ for any $s, u \in \mathcal{T}_n^{ud}(\lambda)$. Note that γ_t can be computed recursively by Rui-Si [17, Proposition 4.9].

2.19. Proposition. *We have the following results:*

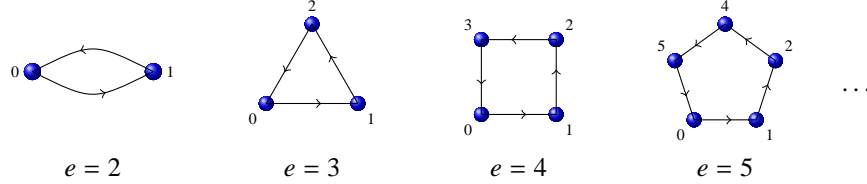
- (1) Suppose $\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$ and $(\lambda, f) \in \widehat{B}_n$. Then $\frac{f_{\mathfrak{t}}}{\gamma_{\mathfrak{t}}}$ is a primitive idempotent of $\mathcal{B}_n^{\overline{\mathbb{R}}}(x)$.
- (2) $\sum_{\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)} \frac{f_{\mathfrak{t}}}{\gamma_{\mathfrak{t}}}$ is a central primitive idempotent. Moreover,

$$\sum_{(\lambda, f) \in \widehat{B}_n} \sum_{\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)} \frac{f_{\mathfrak{t}}}{\gamma_{\mathfrak{t}}} = 1.$$

2.6. Graded cellular algebras $R\mathfrak{S}_n$

In this subsection we will introduce a graded cellular structure on $R\mathfrak{S}_n$ by giving a graded cellular basis of $R\mathfrak{S}_n$.

Khovanov and Lauda [11, 10] and Rouquier [16] have introduced a naturally \mathbb{Z} -graded algebra \mathcal{R}_n . Fix an integer $e \in \{0, 2, 3, 4, \dots\}$. Define $P' = \mathbb{Z} \cup \mathbb{Z}/2$ when $e = 0$ and $P' = \mathbb{Z}/e\mathbb{Z}$ when $e > 0$. Let Γ_e be the oriented quiver with vertex set $\mathbb{Z}/e\mathbb{Z}$ and directed edges $i \rightarrow i+1$, for $i \in \mathbb{Z}/e\mathbb{Z}$. Thus, Γ_e is the quiver of type A_{∞} if $e = 0$ and if $e \geq 2$ then it is a cyclic quiver of type $A_e^{(1)}$:



Let $(a_{i,j})_{i,j \in \mathbb{Z}/e\mathbb{Z}}$ be the symmetric Cartan matrix associated with Γ_e , so that

$$a_{i,j} = \begin{cases} 2, & \text{if } i = j, \\ 0, & \text{if } i \neq j \pm 1, \\ -1, & \text{if } e \neq 2 \text{ and } i = j \pm 1, \\ -2, & \text{if } e = 2 \text{ and } i = j + 1. \end{cases}$$

To the quiver Γ_e attach the standard Lie theoretic data of a Cartan matrix $(a_{ij})_{i,j \in \mathbb{Z}/e\mathbb{Z}}$, fundamental weights $\{\Lambda_i | i \in \mathbb{Z}/e\mathbb{Z}\}$, positive weights $\sum_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{N}\Lambda_i$, positive roots $\bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{N}\alpha_i$ and let (\cdot, \cdot) be the bilinear form determined by

$$(\alpha_i, \alpha_j) = a_{ij} \quad \text{and} \quad (\Lambda_i, \alpha_j) = \delta_{ij}, \quad \text{for } i, j \in \mathbb{Z}/e\mathbb{Z}.$$

Fix a weight $\Lambda = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} a_i \Lambda_i \in \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{N}\Lambda_i$. Then Λ is a weight of level $l(\Lambda) = \ell = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} a_i$. A multicharge for Λ is a sequence $\kappa_{\Lambda} = (\kappa_1, \dots, \kappa_{\ell}) \in (\mathbb{Z}/e\mathbb{Z})^{\ell}$ such that

$$(\Lambda, \alpha_i) = a_i = \# \{ 1 \leq s \leq \ell \mid \kappa_s \equiv i \pmod{e} \}$$

for any $i \in \mathbb{Z}/e\mathbb{Z}$.

The following algebras were introduced by Khovanov and Lauda and Rouquier who defined KLR algebras for arbitrary oriented quivers.

2.20. Definition (Khovanov and Lauda [11, 10] and Rouquier [16]). Suppose \mathcal{K} is an integral ring and n is a positive integer. The *Khovanov-Lauda-Rouquier algebra*, $\mathcal{R}_n(\mathcal{K})$ of type Γ_e is the unital associative \mathcal{K} -algebra with generators

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\mathbf{i}) \mid \mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n\}$$

and relations

$$\begin{aligned} e(\mathbf{i})e(\mathbf{j}) &= \delta_{\mathbf{i}\mathbf{j}}e(\mathbf{i}), & \sum_{\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n} e(\mathbf{i}) &= 1, \\ y_r e(\mathbf{i}) &= e(\mathbf{i})y_r, & \psi_r e(\mathbf{i}) &= e(s_r \cdot \mathbf{i})\psi_r, & y_r y_s &= y_s y_r, \\ \psi_r y_s &= y_s \psi_r, & & & \text{if } s \neq r, r+1, \\ \psi_r \psi_s &= \psi_s \psi_r, & & & \text{if } |r-s| > 1, \end{aligned}$$

$$\begin{aligned}
\psi_r y_{r+1} e(\mathbf{i}) &= \begin{cases} (y_r \psi_r + 1) e(\mathbf{i}), & \text{if } i_r = i_{r+1}, \\ y_r \psi_r e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases} \\
y_{r+1} \psi_r e(\mathbf{i}) &= \begin{cases} (\psi_r y_r + 1) e(\mathbf{i}), & \text{if } i_r = i_{r+1}, \\ \psi_r y_r e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases} \\
\psi_r^2 e(\mathbf{i}) &= \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ e(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \pm 1, \\ (y_{r+1} - y_r) e(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r + 1, \\ (y_r - y_{r+1}) e(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r - 1, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(\mathbf{i}), & \text{if } e = 2 \text{ and } i_{r+1} = i_r + 1 \end{cases} \\
\psi_r \psi_{r+1} \psi_r e(\mathbf{i}) &= \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) e(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+2} = i_r = i_{r+1} - 1, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1) e(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+2} = i_r = i_{r+1} + 1, \\ (\psi_{r+1} \psi_r \psi_{r+1} + y_r - 2y_{r+1} + y_{r+2}) e(\mathbf{i}), & \text{if } e = 2 \text{ and } i_{r+2} = i_r = i_{r+1} + 1, \\ \psi_{r+1} \psi_r \psi_{r+1} e(\mathbf{i}), & \text{otherwise.} \end{cases}
\end{aligned}$$

for $\mathbf{i}, \mathbf{j} \in (\mathbb{Z}/e\mathbb{Z})^n$ and all admissible r and s . Moreover, $\mathcal{R}_n(\mathcal{O})$ is naturally \mathbb{Z} -graded with degree function determined by

$$\deg e(\mathbf{i}) = 0, \quad \deg y_r = 2 \quad \text{and} \quad \deg \psi_s e(\mathbf{i}) = -a_{i_s, i_{s+1}},$$

for $1 \leq r \leq n$, $1 \leq s < n$ and $\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n$.

Fix a weight $\Lambda = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} a_i \Lambda_i$ with $a_i \in \mathbb{N}$. Let $N_n^\Lambda(\mathcal{K})$ be the two-sided ideal of \mathcal{R}_n generated by the elements $e(\mathbf{i}) y_1^{(\Lambda, \alpha_{i_1})}$, for $\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n$. We define the cyclotomic Khovanov-Lauda-Rouquier algebras, which were introduced by Khovanov and Lauda [11, Section 3.4].

2.21. Definition. The *cyclotomic Khovanov-Lauda-Rouquier algebras* of weight Λ and type Γ_e is the algebra $\mathcal{R}_n^\Lambda(\mathcal{K}) = \mathcal{R}_n(\mathcal{K})/N_n^\Lambda(\mathcal{K})$.

Brundan and Kleshchev [3] proved the remarkable result that when \mathcal{K} is a field of characteristic p , $\mathcal{K} \mathfrak{S}_n \cong \mathcal{R}_n^\Lambda(\mathcal{K})$ when we set $e = p$ and $\Lambda = \Lambda_k$ for any $k \in \mathbb{Z}/e\mathbb{Z}$.

2.22. Theorem (Brundan-Kleshchev [3]). *Suppose \mathcal{K} is a field of characteristic p and $\mathcal{R}_n^\Lambda(\mathcal{K})$ is the cyclotomic Khovanov-Lauda-Rouquier algebra over \mathcal{K} with $e = p$ and $\Lambda = \Lambda_k$ for any $k \in \mathbb{Z}/e\mathbb{Z}$. Then $\mathcal{K} \mathfrak{S}_n \cong \mathcal{R}_n^\Lambda(\mathcal{K})$.*

Murphy [13] constructed the first cellular basis for $\mathcal{K} \mathfrak{S}_n$ which shows that $\mathcal{K} \mathfrak{S}_n$ is a cellular algebra. Hu-Mathas [8] gave a graded cellular basis of $\mathcal{K} \mathfrak{S}_n$ and prove that $\mathcal{K} \mathfrak{S}_n$ is a graded cellular algebra. Next we introduce a graded cellular basis of $\mathcal{K} \mathfrak{S}_n$.

Suppose $\Lambda = \Lambda_k$ for some $k \in \mathbb{Z}/e\mathbb{Z}$ and $\lambda \vdash n$ is a partition of n . For $\mathbf{t} \in \text{Std}(\lambda)$, write $\mathbf{t}(\ell) = (r_\ell, c_\ell)$ for $1 \leq \ell \leq n$. We define the *residue sequence* of \mathbf{t} to be $\mathbf{i} = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n$ where $i_\ell \equiv k + c_\ell - r_\ell \pmod{e}$ for $1 \leq \ell \leq n$.

2.23. Remark. Note we have defined residue sequence for up-down tableau. A standard λ -tableau can be considered as a special case of up-down tableau with shape $(\lambda, 0)$. For $\mathbf{t} \in \text{Std}(\lambda)$, we have two residue sequence $-\mathbf{i} = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n$ by considering \mathbf{t} as a standard tableau, and $\mathbf{j} = (j_1, \dots, j_n) \in P^n$ by considering \mathbf{t} as an up-down tableau. Essentially these two definitions are equivalent when $e = p = 0$. One can see that \mathbf{i} is a "shift" of \mathbf{j} . In more details, for $1 \leq \ell \leq n$, we have $i_\ell = j_\ell - (\frac{\delta-1}{2} - k)$.

Define \mathbf{t}^λ to be the unique standard λ -tableau such that $\mathbf{t}^\lambda \triangleright \mathbf{t}$ for all standard λ -tableau \mathbf{t} and let $\mathbf{i}_\lambda = (i_1, \dots, i_n)$ to be the residue sequence of \mathbf{t}^λ by considering it as a standard tableau. We define $e_\lambda = e(\mathbf{i}_\lambda)$.

Suppose $w \in \mathfrak{S}_n$ with reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_m}$. Define

$$\psi_w = \psi_{i_1} \psi_{i_2} \dots \psi_{i_m} \in \mathcal{R}_n^\Lambda(\mathcal{K}) \quad \text{and} \quad \psi_w^* = \psi_{i_m} \psi_{i_{m-1}} \dots \psi_{i_2} \psi_{i_1} \in \mathcal{R}_n^\Lambda(\mathcal{K}).$$

For any standard tableau \mathbf{t} with shape λ , we define $d(\mathbf{t}) \in \mathfrak{S}_n$ such that $\mathbf{t} = \mathbf{t}^\lambda d(\mathbf{t})$.

2.24. Definition. Suppose $\lambda \vdash n$ and $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$. Define

$$\psi_{\mathbf{st}} = \psi_{d(\mathbf{s})}^* e_\lambda \psi_{d(\mathbf{t})} \in \mathcal{R}_n^\Lambda(\mathcal{K}).$$

2.25. **Theorem** (Hu-Mathas [8, Theorem 5.14]). *Suppose \mathcal{K} is a field. Then*

$$\{ \psi_{\mathbf{st}} \mid \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for } \lambda \vdash n \}$$

is a graded cellular basis of $\mathcal{R}_n^\Lambda(\mathcal{K})$.

2.26. **Remark.** We note that all the results of this subsection were originally proved in the cyclotomic Hecke algebras of type A , $\mathcal{H}_n^\Lambda(\mathcal{K})$, with weight $\Lambda = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} a_i \Lambda_i \in \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{N} \Lambda_i$ instead of the symmetric group algebra, $\mathcal{K} \mathfrak{S}_n$. In this paper we only need the results for $\mathcal{K} \mathfrak{S}_n$. So we restrict all the results to $\mathcal{K} \mathfrak{S}_n$.

3. The graded algebras $\mathcal{G}_n(\delta)$

Let x be an invariant and recall $\mathcal{O} = R[x]_{(x-\delta)} = R[[x-\delta]]$, and $\mathbb{F} = R(x)$. In this section we define a new algebra $\mathcal{G}_n(\delta)$ over R associated with KLR-like relations, which is naturally \mathbb{Z} -graded.

3.1. A categorification of n -tuple $\mathbf{i} \in P^n$

Recall $P = \frac{\delta-1}{2} + \mathbb{Z}$. In this subsection we define a mapping $h_k : P^n \rightarrow \mathbb{Z}$ which separates P^n into three mutually exclusive subsets. Such categorification will be used to determine the relations and the degree of generators of the graded algebra $\mathcal{G}_n(\delta)$.

First we give a proper definition of h_k .

3.1. **Definition.** Suppose $\mathbf{i} = (i_1, i_2, \dots, i_n) \in P^n$ and k is an integer with $1 \leq k \leq n$. We define

$$\begin{aligned} h_k(\mathbf{i}) &:= \delta_{i_k, -\frac{\delta-1}{2}} + \# \{ 1 \leq r \leq k-1 \mid i_r = -i_k \pm 1 \} + 2\# \{ 1 \leq r \leq k-1 \mid i_r = i_k \} \\ &\quad - \delta_{i_k, \frac{\delta-1}{2}} - \# \{ 1 \leq r \leq k-1 \mid i_r = i_k \pm 1 \} - 2\# \{ 1 \leq r \leq k-1 \mid i_r = -i_k \}. \end{aligned}$$

3.2. **Remark.** Suppose $\mathbf{i} = (i_1, \dots, i_n)$ and $1 \leq k \leq n-1$. If $i_{k+1} = i_k$, we have

$$h_{k+1}(\mathbf{i}) = \begin{cases} h_k(\mathbf{i}), & \text{if } i_k = 0, \\ h_k(\mathbf{i}) + 3, & \text{if } i_k = \pm \frac{1}{2}, \\ h_k(\mathbf{i}) + 2, & \text{otherwise;} \end{cases} \quad (3.1)$$

and if $i_{k+1} = -i_k$, we have

$$h_{k+1}(\mathbf{i}) = \begin{cases} -h_k(\mathbf{i}), & \text{if } i_k = 0, \\ -h_k(\mathbf{i}) - 3, & \text{if } i_k = \pm \frac{1}{2}, \\ -h_k(\mathbf{i}) - 2, & \text{otherwise.} \end{cases} \quad (3.2)$$

We will use this result frequently in the rest of this paper.

Given $(\lambda, f) \in \widehat{B}_{k-1}$, the key point of h_k is that it gives us a way to understand the structure of $\mathcal{AR}(\lambda)$. In order to connect h_k and $\mathcal{AR}(\lambda)$, we introduce a Lemma which is first proved by Nazarov [14].

3.3. **Lemma** (Nazarov [14, Lemma 3.8]). *Suppose u is a unknown and \mathbf{t} is an up-down tableau of size n . If for $1 \leq k \leq n-1$ we have $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$, then*

$$\frac{u + (x-1)/2}{u - (x-1)/2} \prod_{r=1}^{k-1} \frac{(u + c_{\mathbf{t}}(r))^2 - 1}{(u - c_{\mathbf{t}}(r))^2 - 1} \frac{(u - c_{\mathbf{t}}(r))^2}{(u + c_{\mathbf{t}}(r))^2} = \sum_{\mathbf{s} \stackrel{k}{\leftarrow} \mathbf{t}} \frac{u + c_{\mathbf{s}}(k)}{u - c_{\mathbf{s}}(k)}. \quad (3.3)$$

Suppose λ is a partition and $\alpha \in \mathcal{AR}(\lambda)$. Define

$$\text{res}_\lambda(\alpha) = \begin{cases} \text{res}(\alpha), & \text{if } \alpha \in \mathcal{A}(\lambda), \\ -\text{res}(\alpha), & \text{if } \alpha \in \mathcal{R}(\lambda), \end{cases}$$

and for $i \in P$, we denote $\mathcal{AR}_\lambda(i) = \{ \alpha \in \mathcal{AR}(\lambda) \mid \text{res}_\lambda(\alpha) = i \}$.

The next Lemma gives the most important property of h_k .

3.4. **Lemma.** *For any $\mathbf{i} \in P^n$ such that $\mathbf{i}|_{k-1}$ is the residue sequence of some up-down tableaux with shape (λ, f) , we have $h_k(\mathbf{i}) = |\mathcal{AR}_\lambda(-i_k)| - |\mathcal{AR}_\lambda(i_k)|$.*

Proof. Suppose $\mathbf{u} \in \mathcal{T}_{k-1}^{ud}(\mathbf{i}|_{k-1})$ with shape (λ, f) . Choose any up-down tableau \mathbf{t} of size $k+1$ with $\mathbf{t}|_{k-1} = \mathbf{u}$ and $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$. By the construction of up-down tableaux, there exists such \mathbf{t} as long as $|\mathcal{AR}(\lambda)| > 0$, which is always true.

Substitute $u = i$ into (3.3). Then we have

$$\frac{i + (x-1)/2}{i - (x-1)/2} \prod_{r=1}^{k-1} \frac{(i + c_t(r))^2 - 1}{(i - c_t(r))^2 - 1} \frac{(i - c_t(r))^2}{(i + c_t(r))^2} = \prod_{s \in s_t^k} \frac{i + c_s(k)}{i - c_s(k)} \in \mathbb{F}.$$

For convenience, write $d = \frac{x-\delta}{2}$. Then we have

$$\prod_{s \in s_t^k} \frac{i + c_s(k)}{i - c_s(k)} = \frac{d^{|\mathcal{A}\mathcal{R}_\lambda(-i)|}}{d^{|\mathcal{A}\mathcal{R}_\lambda(i)|}} v_1,$$

for some v_1 invertible in \mathcal{O} and

$$\frac{i + (x-1)/2}{i - (x-1)/2} \prod_{r=1}^{k-1} \frac{(i + c_t(r))^2 - 1}{(i - c_t(r))^2 - 1} \frac{(i - c_t(r))^2}{(i + c_t(r))^2} = \frac{d^{\delta_{i, -\frac{\delta-1}{2}}}}{d^{\delta_{i, \frac{\delta-1}{2}}}} \frac{d^{\#\{1 \leq r \leq k-1 | i_r = -i \pm 1\}}}{d^{\#\{1 \leq r \leq k-1 | i_r = i \pm 1\}}} \frac{d^{2\#\{1 \leq r \leq k-1 | i_r = i\}}}{d^{2\#\{1 \leq r \leq k-1 | i_r = -i\}}} v_2,$$

for some v_2 invertible in \mathcal{O} .

Hence,

$$\frac{d^{|\mathcal{A}\mathcal{R}_\lambda(-i)|}}{d^{|\mathcal{A}\mathcal{R}_\lambda(i)|}} v_1 = \frac{d^{\delta_{i, -\frac{\delta-1}{2}}}}{d^{\delta_{i, \frac{\delta-1}{2}}}} \frac{d^{\#\{1 \leq r \leq k-1 | i_r = -i \pm 1\}}}{d^{\#\{1 \leq r \leq k-1 | i_r = i \pm 1\}}} \frac{d^{2\#\{1 \leq r \leq k-1 | i_r = i\}}}{d^{2\#\{1 \leq r \leq k-1 | i_r = -i\}}} v_2$$

where v_1, v_2 invertible in \mathcal{O} . Because d is not invertible in \mathcal{O} , we have

$$\begin{aligned} |\mathcal{A}\mathcal{R}_\lambda(-i)| - |\mathcal{A}\mathcal{R}_\lambda(i)| &= \delta_{i, -\frac{\delta-1}{2}} + \#\{1 \leq r \leq k-1 | i_r = -i \pm 1\} + 2\#\{1 \leq r \leq k-1 | i_r = i\} \\ &\quad - \delta_{i, \frac{\delta-1}{2}} - \#\{1 \leq r \leq k-1 | i_r = i \pm 1\} - 2\#\{1 \leq r \leq k-1 | i_r = -i\} = h_k(\mathbf{i}), \end{aligned}$$

which completes the proof. \square

The next Corollary is a special case of Lemma 3.4, which shows the connection between $h_k(\mathbf{i}_t)$ and \mathbf{t} .

3.5. Corollary. *Suppose \mathbf{t} is an up-down tableau of size n and \mathbf{i}_t is the residue sequence of \mathbf{t} . For $1 \leq k \leq n$, let $\mathbf{t}_{k-1} = \lambda$. Then we have $h_k(\mathbf{i}_t) = |\mathcal{A}\mathcal{R}_\lambda(-i_k)| - |\mathcal{A}\mathcal{R}_\lambda(i_k)|$.*

The first application of Lemma 3.4 is that when \mathbf{i} is a residue sequence of some up-down tableaux, the value of $h_k(\mathbf{i})$ is bounded.

3.6. Lemma. *Suppose $\mathbf{i} \in P^n$ and $1 \leq k \leq n$. If \mathbf{i} is the residue sequence of an up-down tableau, we have $h_k(\mathbf{i}) \in \{-2, -1, 0\}$.*

Proof. Suppose \mathbf{t} is an up-down tableau with residue sequence \mathbf{i} . Write $\lambda = \mathbf{t}_{k-1}$. By Lemma 3.4 we have $|\mathcal{A}\mathcal{R}_\lambda(-i_k)| - |\mathcal{A}\mathcal{R}_\lambda(i_k)| = h_k(\mathbf{i})$.

The existence of \mathbf{t} implies $|\mathcal{A}\mathcal{R}_\lambda(i_k)| \geq 1$. By the construction of partitions, we have

$$0 \leq |\mathcal{A}\mathcal{R}_\lambda(-i_k)| \leq 2, \quad 0 \leq |\mathcal{A}\mathcal{R}_\lambda(i_k)| \leq 2, \quad \text{and} \quad 0 \leq |\mathcal{A}\mathcal{R}_\lambda(-i_k)| + |\mathcal{A}\mathcal{R}_\lambda(i_k)| \leq 2. \quad (3.4)$$

The Lemma follows easily by direct calculations. \square

3.7. Lemma. *Suppose $\mathbf{i} \in P^n$ and $1 \leq k \leq n$. If \mathbf{i} is the residue sequence of an up-down tableau, we have $h_k(\mathbf{i}) = 0$ if $i_k = 0$ and $h_k(\mathbf{i}) \in \{-1, -2\}$ if $i_k = \pm \frac{1}{2}$.*

Proof. Suppose $i_k = 0$. As $i_k = -i_k$, by the definition of $h_k(\mathbf{i})$, we have $h_k(\mathbf{i}) = -h_k(\mathbf{i}) = 0$. Suppose $i_k = -\frac{1}{2}$ and set $\lambda = \mathbf{t}_{k-1}$. By the construction of λ , we have $|\mathcal{A}\mathcal{R}_\lambda(-i_k)| = 0$ and $|\mathcal{A}\mathcal{R}_\lambda(i_k)| \geq 1$, which implies that $h_k(\mathbf{i}) \leq -1$ by Lemma 3.4. Hence $h_k(\mathbf{i}) \in \{-1, -2\}$. For $i_k = \frac{1}{2}$ we have the same result following the same argument. \square

Lemma 3.6 and Lemma 3.7 give us an easy way to test whether $\mathbf{i} \in P^n$ is the residue sequence of an up-down tableau. If we have $h_k(\mathbf{i}) \notin \{-2, -1, 0\}$ for some $1 \leq k \leq n$, then \mathbf{i} is not the residue sequence of an up-down tableau. But the reverse is not always valid. We will discuss this problem further in Section 3.3.

The next important application of h_k is by giving $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and $\lambda = \mathbf{t}_{k-1}$ for $1 \leq k \leq n$, we know the exact values of $|\mathcal{A}\mathcal{R}_\lambda(-i_k)|$ and $|\mathcal{A}\mathcal{R}_\lambda(i_k)|$ by knowing $h_k(\mathbf{i})$. Because \mathbf{i} is the residue sequence of \mathbf{t} , we have $|\mathcal{A}\mathcal{R}_\lambda(i_k)| \geq 1$. Then by Lemma 3.4 and (3.4), the following results are straightforward:

$$|\mathcal{A}\mathcal{R}_\lambda(-i_k)| = 0, \quad |\mathcal{A}\mathcal{R}_\lambda(i_k)| = 2 \quad \text{if } h_k(\mathbf{i}) = -2, \quad (3.5)$$

$$|\mathcal{A}\mathcal{R}_\lambda(-i_k)| = 0, \quad |\mathcal{A}\mathcal{R}_\lambda(i_k)| = 1 \quad \text{if } h_k(\mathbf{i}) = -1, \quad (3.6)$$

$$|\mathcal{A}\mathcal{R}_\lambda(-i_k)| = 1, \quad |\mathcal{A}\mathcal{R}_\lambda(i_k)| = 1 \quad \text{if } h_k(\mathbf{i}) = 0, \quad (3.7)$$

The following results are implied by (3.5) - (3.7), which can be used to determine the structure of \mathbf{t} . These results will be used frequently in the rest of this paper.

3.8. Lemma. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. For $1 \leq k \leq n$, write $\lambda = \mathbf{t}_{k-1}$. Then we have the following properties:

- (1) When $h_k(\mathbf{i}) = -2$, then $\mathcal{AR}_\lambda(i_k) = \{\alpha, \beta\}$ where $\alpha \in \mathcal{A}(\lambda)$ and $\beta \in \mathcal{R}(\lambda)$.
- (2) When $h_k(\mathbf{i}) = 0$, then $\mathcal{AR}_\lambda(i_k) = \{\alpha\}$ and $\mathcal{AR}_\lambda(-i_k) = \{\beta\}$, where either $\alpha, \beta \in \mathcal{A}(\lambda)$ or $\alpha, \beta \in \mathcal{R}(\lambda)$.

Proof. (1). When $h_k(\mathbf{i}) = -2$, by (3.5) we have $\mathcal{AR}_\lambda(i_k) = \{\alpha, \beta\}$. Suppose $\alpha \in \mathcal{A}(\lambda)$. If $\beta \in \mathcal{A}(\lambda)$, then we have $\alpha, \beta \in \mathcal{A}(\lambda)$ such that $\text{res}(\alpha) = \text{res}(\beta) = i_k$. But for any λ , there exists at most one addable node with residue i_k . Hence we must have $\beta \in \mathcal{R}(\lambda)$. Suppose $\alpha \in \mathcal{R}(\lambda)$. If $\beta \in \mathcal{R}(\lambda)$, then we have $\alpha, \beta \in \mathcal{R}(\lambda)$ such that $\text{res}(\alpha) = \text{res}(\beta) = -i_k$. But for any λ , there exists at most one removable node with residue $-i_k$. Hence we must have $\beta \in \mathcal{A}(\lambda)$. Therefore part (1) follows.

(2). When $h_k(\mathbf{i}) = 0$, by (3.7) we have $\mathcal{AR}_\lambda(i_k) = \{\alpha\}$ and $\mathcal{AR}_\lambda(-i_k) = \{\beta\}$. Suppose $\alpha \in \mathcal{A}(\lambda)$. If $\beta \in \mathcal{R}(\lambda)$, we have $\text{res}(\alpha) = \text{res}(\beta) = i_k$. But for any λ , if there exists an addable node with residue i_k , there does not exist a removable node with residue i_k . Hence we must have $\beta \in \mathcal{A}(\lambda)$. Suppose $\alpha \in \mathcal{R}(\lambda)$. If $\beta \in \mathcal{A}(\lambda)$, we have $\text{res}(\alpha) = \text{res}(\beta) = -i_k$. But for any λ , if there exists an addable node with residue $-i_k$, there does not exist a removable node with residue $-i_k$. Hence we must have $\beta \in \mathcal{R}(\lambda)$. Therefore part (2) follows. \square

3.9. Lemma. Suppose $\mathbf{i} \in P^n$ with $i_k = i_{k+1}$ for $1 \leq k \leq n-1$. For $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $\mathbf{t}(k) > 0$, $\mathbf{t}(k+1) < 0$ or $\mathbf{t}(k) < 0$, $\mathbf{t}(k+1) > 0$. Moreover, we have $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$ if and only if $i_k = 0$.

Proof. Because of (3.1) and Lemma 3.6, it forces $i_k \neq \pm \frac{1}{2}$, and

$$h_k(\mathbf{i}) = \begin{cases} 0, & \text{if } i_k = 0, \\ -2, & \text{if } i_k \neq 0, \end{cases} \quad \text{and } h_{k+1}(\mathbf{i}) = 0.$$

Write $\lambda = \mathbf{t}_{k-1}$. Assume $i_k = 0$. By (3.7) we have $|\mathcal{AR}_\lambda(i_k)| = 1$. Hence by the construction of up-down tableaux, we require $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$, which also implies that $\mathbf{t}(k) > 0$, $\mathbf{t}(k+1) < 0$ or $\mathbf{t}(k) < 0$, $\mathbf{t}(k+1) > 0$.

Assume $i_k \neq 0$. By (3.5) we have $|\mathcal{AR}_\lambda(i_k)| = 2$. Let $\alpha, \beta \in \mathcal{AR}_\lambda(i_k)$ be distinct nodes. By Lemma 3.8, we set $\alpha \in \mathcal{A}(\lambda)$ and $\beta \in \mathcal{R}(\lambda)$. By the construction of up-down tableaux, we require $\mathbf{t}(k) = \alpha$, $\mathbf{t}(k+1) = -\beta$ or $\mathbf{t}(k) = -\beta$, $\mathbf{t}(k+1) = \alpha$. Henceforth, we have $\mathbf{t}(k) > 0$, $\mathbf{t}(k+1) < 0$ or $\mathbf{t}(k) < 0$, $\mathbf{t}(k+1) > 0$. Because α and β are distinct, we have $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$, which completes the proof. \square

3.10. Lemma. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. If $\mathbf{t}_{k-1} = \mathbf{t}_{k+1} = \lambda$ for $1 \leq k \leq n-1$, we have the following properties:

- (1) $h_k(\mathbf{i}) = -2$ if and only if there exists a unique $\mathbf{s} \neq \mathbf{t}$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$ and $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})$. Moreover, we have $c_t(k) - i_k = -(c_s(k) - i_{k+1})$.
- (2) $h_k(\mathbf{i}) = 0$ if and only if there exists a unique \mathbf{s} such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$ and $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$. Moreover, we have $c_t(k) - i_k = c_s(k) - i_k$.
- (3) $h_k(\mathbf{i}) = -1$ if and only if $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i}) \cup \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ and $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$ implies $\mathbf{s} = \mathbf{t}$.

Proof. In (1), by (3.5) we have $|\mathcal{AR}_\lambda(-i_k)| = 0$ and $|\mathcal{AR}_\lambda(i_k)| = 2$ if and only if $h_k(\mathbf{i}) = -2$. Hence by Lemma 2.17, $|\mathcal{AR}_\lambda(i_k)| = 2$ if and only if there exist exactly two distinct up-down tableaux $\mathbf{u}, \mathbf{v} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{u} \stackrel{k}{\sim} \mathbf{t}$ and $\mathbf{v} \stackrel{k}{\sim} \mathbf{t}$. It is obvious that one of \mathbf{u} and \mathbf{v} is \mathbf{t} . It is Without loss of generality, we set $\mathbf{u} = \mathbf{t}$. Hence by setting $\mathbf{s} = \mathbf{v}$, the uniqueness and existence of \mathbf{s} follows.

Moreover, by Lemma 3.8 we have $\mathcal{AR}_\lambda(i_k) = \{\alpha, \beta\}$ where $\alpha \in \mathcal{A}(\lambda)$ and $\beta \in \mathcal{R}(\lambda)$. Then we have $\mathbf{t}(k) = \alpha$ and $\mathbf{s}(k) = -\beta$, or vice versa. In both cases, we have $c_t(k) - i_k = -(c_s(k) - i_{k+1})$, which proves part (1).

Using similar arguments, (2) can be implied by (3.7) and Lemma 3.8; and (3) can be implied by (3.6). \square

3.11. Lemma. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. If $i_k + i_{k+1} = 0$ for $1 \leq k \leq n-1$, we have the following properties:

- (1) When $h_{k+1}(\mathbf{i}) = 0$ or -1 , then $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$.
- (2) When $h_{k+1}(\mathbf{i}) = -2$, then either $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$, or $c_t(k) - i_k = c_t(k+1) - i_{k+1}$.

Proof. Suppose $\mathbf{t}_k = \lambda$ and $\mathbf{t}(k) = \alpha$. Without loss of generality, we assume $\alpha > 0$. When $\alpha < 0$ the Lemma follows by the same argument.

(1). When $h_{k+1}(\mathbf{i}) = 0$ or -1 , by (3.6) and (3.7) we have $|\mathcal{AR}_\lambda(i_{k+1})| = 1$. As $\alpha \in \mathcal{R}(\lambda)$ and $\text{res}_\lambda(\alpha) = -\text{res}(\alpha) = -i_k = i_{k+1}$, we have $\mathcal{AR}_\lambda(i_{k+1}) = \{\alpha\}$. Hence, it forces $\mathbf{t}(k+1) = -\alpha = -\mathbf{t}(k)$.

(2). When $h_{k+1}(\mathbf{i}) = -2$, by (3.5) we have $|\mathcal{AR}_\lambda(i_{k+1})| = 2$. For the same reason as above, we have $\alpha \in \mathcal{AR}_\lambda(i_{k+1})$. Hence we have $\mathcal{AR}_\lambda(i_{k+1}) = \{\alpha, \beta\}$. By Lemma 3.8, we have $\alpha \in \mathcal{R}(\lambda)$ and $\beta \in \mathcal{A}(\lambda)$. Therefore $\mathbf{t}(k+1) = -\alpha$ or β . If $\mathbf{t}(k+1) = -\alpha = -\mathbf{t}(k)$, we have $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$; and if $\mathbf{t}(k+1) = \beta$, we have $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$ and $c_t(k) - i_k = \frac{x-\delta}{2} = c_t(k+1) - i_{k+1}$. \square

We now categorize P^n using h_k . For $1 \leq k \leq n$, define $P_{k,+}^n$, $P_{k,-}^n$ and $P_{k,0}^n$ as subsets of P^n by

$$\begin{aligned} P_{k,+}^n &:= \{ \mathbf{i} \in P^n \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -1 \}, \\ P_{k,-}^n &:= \{ \mathbf{i} \in P^n \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -2, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -3 \}, \\ P_{k,0}^n &:= P^n \setminus (P_{k,+}^n \cup P_{k,-}^n). \end{aligned}$$

We split P^n into three mutually exclusive subsets, i.e. $P^n = P_{k,+}^n \sqcup P_{k,-}^n \sqcup P_{k,0}^n$. Let I^n be the set containing all the residue sequences of up-down tableaux of size n . For $1 \leq k \leq n$, define

$$I_{k,a}^n := \{ \mathbf{i} \in P_{k,a}^n \mid \mathbf{i} \text{ is a residue sequence of some up-down tableaux} \}$$

where $a \in \{+, -, 0\}$. It is easy to see that $I^n = I_{k,+}^n \sqcup I_{k,-}^n \sqcup I_{k,0}^n$. By the definitions of $P_{k,+}^n$, $P_{k,-}^n$ and $P_{k,0}^n$, Lemma 3.6 and Lemma 3.7 imply that

$$\begin{aligned} I_{k,+}^n &= \{ \mathbf{i} \in I^n \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -1 \}, \\ I_{k,-}^n &= \{ \mathbf{i} \in I^n \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -2 \}, \\ I_{k,0}^n &= \{ \mathbf{i} \in I^n \mid i_k \neq 0, -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -1, \text{ or } i_k = -\frac{1}{2} \text{ and } h_k(\mathbf{i}) = -2, \text{ or } i_k = 0 \}. \end{aligned}$$

We will give a further explanation of this categorification after we construct our graded algebra $\mathcal{G}_n(\delta)$.

3.2. Graded algebras $\mathcal{G}_n(\delta)$

In this subsection we construct a naturally \mathbb{Z} -graded algebra $\mathcal{G}_n(\delta)$ over R and introduce some of its properties. For $\mathbf{i} \in P^n$ and $1 \leq k \leq n-1$, define $a_k(\mathbf{i}) \in \mathbb{Z}$ and $A_{k,1}^{\mathbf{i}}, A_{k,2}^{\mathbf{i}}, A_{k,3}^{\mathbf{i}}, A_{k,4}^{\mathbf{i}} \in \{1, 2, \dots, k-1\}$ by

$$a_k(\mathbf{i}) = \begin{cases} \# \{ 1 \leq r \leq k-1 \mid i_r \in \{-1, 1\} \} + 1 + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } \frac{i_k - i_{k+1}}{2} = 0, \\ \# \{ 1 \leq r \leq k-1 \mid i_r \in \{-1, 1\} \} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } \frac{i_k - i_{k+1}}{2} = 1, \\ \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } \frac{i_k - i_{k+1}}{2} = 1/2, \\ \# \{ 1 \leq r \leq k-1 \mid i_r \in \{ \frac{i_k - i_{k+1}}{2}, \frac{i_k - i_{k+1}}{2} - 1, -\frac{i_k - i_{k+1}}{2}, -\frac{i_k - i_{k+1}}{2} + 1 \} \} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} A_{k,1}^{\mathbf{i}} &:= \{ 1 \leq r \leq k-1 \mid i_r = -i_k \pm 1 \}, & A_{k,2}^{\mathbf{i}} &:= \{ 1 \leq r \leq k-1 \mid i_r = i_k \}, \\ A_{k,3}^{\mathbf{i}} &:= \{ 1 \leq r \leq k-1 \mid i_r = i_k \pm 1 \}, & A_{k,4}^{\mathbf{i}} &:= \{ 1 \leq r \leq k-1 \mid i_r = -i_k \}; \end{aligned}$$

and for $\mathbf{i} \in P_{k,0}^n$ and $1 \leq k \leq n-1$, define $z_k(\mathbf{i}) \in \mathbb{Z}$ by

$$z_k(\mathbf{i}) = \begin{cases} 0, & \text{if } h_k(\mathbf{i}) < -2, \text{ or } h_k(\mathbf{i}) \geq 0 \text{ and } i_k \neq 0, \\ (-1)^{a_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}), & \text{if } -2 \leq h_k(\mathbf{i}) < 0, \\ \frac{1 + (-1)^{a_k(\mathbf{i})}}{2}, & \text{if } i_k = 0. \end{cases}$$

Let $\mathcal{G}_n(\delta)$ be an unital associate R -algebra with generators

$$G_n(\delta) = \{ e(\mathbf{i}) \mid \mathbf{i} \in P^n \} \cup \{ y_k \mid 1 \leq k \leq n \} \cup \{ \psi_k \mid 1 \leq k \leq n-1 \} \cup \{ \epsilon_k \mid 1 \leq k \leq n-1 \}$$

associated with the following relations:

(1) Idempotent relations: Let $\mathbf{i}, \mathbf{j} \in P^n$ and $1 \leq k \leq n-1$. Then

$$y_1^{\delta_{i_1, \frac{\delta-1}{2}}} e(\mathbf{i}) = 0, \quad \sum_{\mathbf{i} \in P^n} e(\mathbf{i}) = 1, \quad e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i}, \mathbf{j}} e(\mathbf{i}), \quad e(\mathbf{i})\epsilon_k = 0 \text{ if } i_k + i_{k+1} \neq 0; \quad (3.8)$$

(2) Commutation relations: Let $\mathbf{i} \in P^n$. Then

$$y_k e(\mathbf{i}) = e(\mathbf{i}) y_k, \quad \psi_k e(\mathbf{i}) = e(\mathbf{i} \cdot s_k) \psi_k \quad \text{and} \quad (3.9)$$

$$y_k y_r = y_r y_k, \quad y_k \psi_r = \psi_r y_k, \quad y_k \epsilon_r = \epsilon_r y_k, \quad (3.10)$$

$$\psi_k \psi_r = \psi_r \psi_k, \quad \psi_k \epsilon_r = \epsilon_r \psi_k, \quad \epsilon_k \epsilon_r = \epsilon_r \epsilon_k \quad \text{if } |k-r| > 1; \quad (3.11)$$

(3) Essential commutation relations: Let $\mathbf{i} \in P^n$ and $1 \leq k \leq n-1$. Then

$$e(\mathbf{i}) y_k \psi_k = e(\mathbf{i}) \psi_k y_{k+1} + e(\mathbf{i}) \epsilon_k e(\mathbf{i} \cdot s_k) - \delta_{i_k, i_{k+1}} e(\mathbf{i}), \quad (3.12)$$

$$\text{and} \quad e(\mathbf{i}) \psi_k y_k = e(\mathbf{i}) y_{k+1} \psi_k + e(\mathbf{i}) \epsilon_k e(\mathbf{i} \cdot s_k) - \delta_{i_k, i_{k+1}} e(\mathbf{i}). \quad (3.13)$$

(4) Inverse relations: Let $\mathbf{i} \in P^n$ and $1 \leq k \leq n-1$. Then

$$e(\mathbf{i})\psi_k^2 = \begin{cases} 0, & \text{if } i_k = i_{k+1} \text{ or } i_k + i_{k+1} = 0 \text{ and } h_k(\mathbf{i}) \neq 0, \\ (y_k - y_{k+1})e(\mathbf{i}), & \text{if } i_k = i_{k+1} + 1 \text{ and } i_k + i_{k+1} \neq 0, \\ (y_{k+1} - y_k)e(\mathbf{i}), & \text{if } i_k = i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0, \\ e(\mathbf{i}), & \text{otherwise;} \end{cases} \quad (3.14)$$

(5) Essential idempotent relations: Let $\mathbf{i}, \mathbf{j}, \mathbf{k} \in P^n$ and $1 \leq k \leq n-1$. Then

$$e(\mathbf{i})\epsilon_k e(\mathbf{i}) = \begin{cases} (-1)^{a_k(\mathbf{i})}e(\mathbf{i}), & \text{if } \mathbf{i} \in P_{k,0}^n \text{ and } i_k = -i_{k+1} \neq \pm \frac{1}{2}, \\ (-1)^{a_k(\mathbf{i})+1}(y_{k+1} - y_k)e(\mathbf{i}), & \text{if } \mathbf{i} \in P_{k,+}^n; \end{cases} \quad (3.15)$$

$$y_{k+1}e(\mathbf{i}) = y_k e(\mathbf{i}) - 2(-1)^{a_k(\mathbf{i})}y_k e(\mathbf{i})\epsilon_k e(\mathbf{i}) \quad (3.16)$$

$$= y_k e(\mathbf{i}) - 2(-1)^{a_k(\mathbf{i})}e(\mathbf{i})\epsilon_k e(\mathbf{i})y_k, \quad \text{if } \mathbf{i} \in P_{k,0}^n \text{ and } i_k = -i_{k+1} = \frac{1}{2}, \quad (3.17)$$

$$\begin{aligned} e(\mathbf{i}) &= (-1)^{a_k(\mathbf{i})}e(\mathbf{i})\epsilon_k e(\mathbf{i}) - 2(-1)^{a_{k-1}(\mathbf{i})}e(\mathbf{i})\epsilon_{k-1}e(\mathbf{i}) \\ &\quad + e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}) + e(\mathbf{i})\epsilon_k \epsilon_{k-1}e(\mathbf{i}), \quad \text{if } \mathbf{i} \in P_{k,0}^n \text{ and } -i_{k-1} = i_k = -i_{k+1} = -\frac{1}{2}, \end{aligned} \quad (3.18)$$

$$e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})}e(\mathbf{i})(\epsilon_k y_k + y_k \epsilon_k)e(\mathbf{i}), \quad \text{if } \mathbf{i} \in P_{k,-}^n \text{ and } i_k = -i_{k+1}, \quad (3.19)$$

$$e(\mathbf{j})\epsilon_k e(\mathbf{i})\epsilon_k e(\mathbf{k}) = \begin{cases} z_k(\mathbf{i})e(\mathbf{j})\epsilon_k e(\mathbf{k}), & \text{if } \mathbf{i} \in P_{k,0}^n, \\ 0, & \text{if } \mathbf{i} \in P_{k,-}^n, \\ (-1)^{a_k(\mathbf{i})}(1 + \delta_{i_k, -\frac{1}{2}})(\sum_{r \in A_{k,1}^{\mathbf{i}}} y_r - 2 \sum_{r \in A_{k,2}^{\mathbf{i}}} y_r, \\ \quad + \sum_{r \in A_{k,3}^{\mathbf{i}}} y_r - 2 \sum_{r \in A_{k,4}^{\mathbf{i}}} y_r)e(\mathbf{j})\epsilon_k e(\mathbf{k}), & \text{if } \mathbf{i} \in P_{k,+}^n; \end{cases} \quad (3.20)$$

(6) Untwist relations: Let $\mathbf{i}, \mathbf{j} \in P^n$ and $1 \leq k \leq n-1$. Then

$$e(\mathbf{i})\psi_k \epsilon_k e(\mathbf{j}) = \begin{cases} (-1)^{a_k(\mathbf{i})}e(\mathbf{i})\epsilon_k e(\mathbf{j}), & \text{if } \mathbf{i} \in P_{k,+}^n \text{ and } i_k \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases} \quad (3.21)$$

$$e(\mathbf{j})\epsilon_k \psi_k e(\mathbf{i}) = \begin{cases} (-1)^{a_k(\mathbf{i})}e(\mathbf{j})\epsilon_k e(\mathbf{i}), & \text{if } \mathbf{i} \in P_{k,+}^n \text{ and } i_k \neq 0, -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases} \quad (3.22)$$

(7) Tangle relations: Let $\mathbf{i}, \mathbf{j} \in P^n$ and $1 < k < n$. Then

$$e(\mathbf{j})\epsilon_k \epsilon_{k-1} \psi_k e(\mathbf{i}) = e(\mathbf{j})\epsilon_k \psi_{k-1} e(\mathbf{i}), \quad e(\mathbf{i})\psi_k \epsilon_{k-1} \epsilon_k e(\mathbf{j}) = e(\mathbf{i})\psi_{k-1} \epsilon_k e(\mathbf{j}), \quad (3.23)$$

$$e(\mathbf{i})\epsilon_k \epsilon_{k-1} \epsilon_k e(\mathbf{j}) = e(\mathbf{i})\epsilon_k e(\mathbf{j}); \quad e(\mathbf{i})\epsilon_{k-1} \epsilon_k \epsilon_{k-1} e(\mathbf{j}) = e(\mathbf{i})\epsilon_{k-1} e(\mathbf{j}); \quad e(\mathbf{i})\epsilon_k e(\mathbf{j})(y_k + y_{k+1}) = 0; \quad (3.24)$$

(8) Braid relations: Let $\mathcal{B}_k = \psi_k \psi_{k-1} \psi_k - \psi_{k-1} \psi_k \psi_{k-1}$, $\mathbf{i} \in P^n$ and $1 < k < n$. Then

$$e(\mathbf{i})\epsilon_k \epsilon_{k-1} e(\mathbf{i} \cdot s_k s_{k-1} s_k), \quad \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k - 1), \quad (3.25)$$

$$-e(\mathbf{i})\epsilon_k \epsilon_{k-1} e(\mathbf{i} \cdot s_k s_{k-1} s_k), \quad \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k + 1), \quad (3.26)$$

$$e(\mathbf{i})\epsilon_{k-1} \epsilon_k e(\mathbf{i} \cdot s_k s_{k-1} s_k), \quad \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k - 1), \quad (3.27)$$

$$-e(\mathbf{i})\epsilon_{k-1} \epsilon_k e(\mathbf{i} \cdot s_k s_{k-1} s_k), \quad \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k + 1), \quad (3.28)$$

$$-(-1)^{a_{k-1}(\mathbf{i})}e(\mathbf{i})\epsilon_{k-1} e(\mathbf{i} \cdot s_k s_{k-1} s_k), \quad \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm \frac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \quad (3.29)$$

$$(-1)^{a_k(\mathbf{i})}e(\mathbf{i})\epsilon_k e(\mathbf{i} \cdot s_k s_{k-1} s_k), \quad \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm \frac{1}{2} \text{ and } h_{k-1}(\mathbf{i}) = 0, \quad (3.30)$$

$$\begin{aligned} e(\mathbf{i}), & \quad \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \quad \text{and } i_{k-1} = i_{k+1} = i_k - 1, \end{aligned} \quad (3.31)$$

$$\begin{aligned} -e(\mathbf{i}), & \quad \text{if } i_{k-1} + i_k, i_{k-1} + i_{k+1}, i_k + i_{k+1} \neq 0 \\ & \quad \text{and } i_{k-1} = i_{k+1} = i_k + 1, \end{aligned} \quad (3.32)$$

$$0, \quad \text{otherwise.} \quad (3.33)$$

The algebra is self-graded, where the degree of $e(\mathbf{i})$ is 0, y_k is 2 and

$$\deg e(\mathbf{i})\psi_k = \begin{cases} 1, & \text{if } i_k = i_{k+1} \pm 1, \\ -2, & \text{if } i_k = i_{k+1}, \\ 0, & \text{otherwise;} \end{cases}$$

$$\deg_k(\mathbf{i}) = \begin{cases} 1, & \text{if } \mathbf{i} \in P_{k,+}^n, \\ -1, & \text{if } \mathbf{i} \in P_{k,-}^n, \\ 0, & \text{if } \mathbf{i} \in P_{k,0}^n. \end{cases}$$
$$\begin{aligned}
e(\mathbf{i}) &= \begin{array}{c} i_1 \quad i_2 \quad \dots \quad i_n \\ \left| \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \right| \\ \text{---} \text{---} \text{---} \end{array}, \\
\psi_r e(\mathbf{i}) &= \begin{array}{c} i_1 \quad \dots \quad i_{r-1} \quad i_r \quad i_{r+1} \quad i_{r+2} \quad \dots \quad i_n \\ \left| \quad \begin{array}{c} \text{---} \text{---} \end{array} \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad \begin{array}{c} \text{---} \text{---} \end{array} \right| \\ \text{---} \text{---} \end{array}, \\
y_s e(\mathbf{i}) &= \begin{array}{c} i_1 \quad \dots \quad i_{s-1} \quad i_s \quad i_{s+1} \quad \dots \quad i_n \\ \left| \quad \begin{array}{c} \text{---} \text{---} \end{array} \quad \begin{array}{c} \bullet \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \text{---} \end{array} \right| \\ \text{---} \text{---} \end{array}, \\
\epsilon_r e(\mathbf{i}) &= \begin{array}{c} i_1 \quad \dots \quad i_{r-1} \quad i_r \quad i_{r+1} \quad i_{r+2} \quad \dots \quad i_n \\ \left| \quad \begin{array}{c} \text{---} \text{---} \end{array} \quad \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \begin{array}{c} \text{---} \text{---} \end{array} \right| \\ \text{---} \text{---} \end{array},
\end{aligned}$$

Proof. We prove the Lemma by induction. When $n = 1$, the Lemma follows trivially. Assume when $n' < n$ the Lemma holds. When $n' = n$, set $\mathbf{j} = (i_1, \dots, i_{n-1})$. By induction, \mathbf{j} is the residue sequence of some up-down tableau. Suppose \mathbf{u} is the up-down tableau with residue sequence \mathbf{j} and $(\lambda, f) = \text{Shape}(\mathbf{u})$. As $h_n(\mathbf{i}) \in \{-2, -1\}$, we have $|\mathcal{AR}_\lambda(i_n)| > 0$ by Lemma 3.4. Let $\alpha \in \mathcal{AR}_\lambda(i_n)$. Without loss of generality we assume $\alpha \in \mathcal{A}(\lambda)$. Therefore if we write $\mathbf{u} = (\alpha_1, \dots, \alpha_{n-1})$, then $\mathbf{s} = (\alpha_1, \dots, \alpha_{n-1}, \alpha)$ is an up-down tableau with residue sequence \mathbf{i} . \square

If we have $h_k(\mathbf{i}) = 0$ for some $1 \leq k \leq n$, by Lemma 3.4 and (3.4) we have $|\mathcal{AR}_\lambda(-i_k)| = |\mathcal{AR}_\lambda(i_k)| \in \{0, 1\}$. Hence we cannot decide whether \mathbf{i} is the residue sequence of some up-down tableau. Note that when $i_k = 0$, we have $h_k(\mathbf{i}) = 0$. In the rest of this subsection we extend Lemma 3.13 and include the case when $i_k = 0$.

3.14. Lemma. *Suppose $\mathbf{i} \in P^n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ is an up-down tableau with residue sequence \mathbf{i} . Then we have*

$$\begin{aligned} & \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} \\ &= \# \{ 1 \leq k \leq n \mid i_k = 1 \} - \# \{ 1 \leq k \leq n \mid i_k = -1 \}. \end{aligned}$$

Proof. Apply induction on n . The base case $n = 1$ follows trivially. For the induction step, we assume that the Lemma holds for all $\mathbf{j} \in P^{n-1}$ and prove the Lemma holds for $\mathbf{i} \in P^n$. For convenience we denote

$$\begin{aligned} g_1(\lambda) &= \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \}, & g_{-1}(\lambda) &= \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} \\ n_1(\mathbf{i}) &= \# \{ 1 \leq k \leq n \mid i_k = 1 \}, & n_{-1}(\mathbf{i}) &= \# \{ 1 \leq k \leq n \mid i_k = -1 \}, \end{aligned}$$

and we need to prove that

$$g_1(\lambda) - g_{-1}(\lambda) = n_1(\mathbf{i}) - n_{-1}(\mathbf{i}). \quad (3.42)$$

Let $\mathbf{j} = (i_1, \dots, i_{n-1})$ and $\mathbf{s} = \mathbf{t}_{|n-1} \in \mathcal{T}_{n-1}^{ud}(\mu)$ for some μ . We note that if $\mathbf{t}(n) = \alpha > 0$, we have $\lambda = \mu \cup \{\alpha\}$; and if $\mathbf{t}(n) = -\alpha < 0$, we have $\mu = \lambda \cup \{\alpha\}$.

By the construction, \mathbf{s} is an up-down tableau with residue sequence \mathbf{j} . By induction we have

$$g_1(\mu) - g_{-1}(\mu) = n_1(\mathbf{j}) - n_{-1}(\mathbf{j}). \quad (3.43)$$

If $i_n \neq \pm 1$, we have $g_1(\lambda) = g_1(\mu)$, $g_{-1}(\lambda) = g_{-1}(\mu)$, $n_1(\mathbf{i}) = n_1(\mathbf{j})$ and $n_{-1}(\mathbf{i}) = n_{-1}(\mathbf{j})$. Hence (3.42) holds by (3.43).

If $i_n = 1$ and $\mathbf{t}(n) > 0$, we have $g_1(\lambda) = g_1(\mu) + 1$, $g_{-1}(\lambda) = g_{-1}(\mu)$, $n_1(\mathbf{i}) = n_1(\mathbf{j}) + 1$ and $n_{-1}(\mathbf{i}) = n_{-1}(\mathbf{j})$; and if $i_n = 1$ and $\mathbf{t}(n) < 0$, we have $g_1(\lambda) = g_1(\mu)$, $g_{-1}(\lambda) = g_{-1}(\mu) - 1$, $n_1(\mathbf{i}) = n_1(\mathbf{j}) + 1$ and $n_{-1}(\mathbf{i}) = n_{-1}(\mathbf{j})$. Hence (3.42) holds by (3.43).

If $i_n = -1$ and $\mathbf{t}(n) > 0$, we have $g_1(\lambda) = g_1(\mu)$, $g_{-1}(\lambda) = g_{-1}(\mu) + 1$, $n_1(\mathbf{i}) = n_1(\mathbf{j})$ and $n_{-1}(\mathbf{i}) = n_{-1}(\mathbf{j}) + 1$; and if $i_n = -1$ and $\mathbf{t}(n) < 0$, we have $g_1(\lambda) = g_1(\mu) - 1$, $g_{-1}(\lambda) = g_{-1}(\mu)$, $n_1(\mathbf{i}) = n_1(\mathbf{j})$ and $n_{-1}(\mathbf{i}) = n_{-1}(\mathbf{j}) + 1$. Hence (3.42) holds by (3.43). \square

One can see that by knowing the values of δ and $\# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \}$, we can determine the value of $|\mathcal{AR}_\lambda(0)|$.

3.15. Lemma. *Suppose $(\lambda, f) \in \widehat{B}_n$. Then $|\mathcal{AR}_\lambda(0)| = 1$ if and only if δ is odd and one of the following conditions holds:*

- (1) $\delta = 1$ and $\# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} = 0$;
- (2) $\delta < 1$ and $\# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} = -1$;
- (3) $\delta > 1$ and $\# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} = 1$.

Proof. The Lemma follows directly by the construction of $[\lambda]$. \square

The next result is directly implied by Lemma 3.14 and Lemma 3.15. Define

$$a_k^*(\mathbf{i}) = \# \{ 1 \leq r \leq k-1 \mid i_r \in \{-1, 1\} \} + \delta_{0, \frac{\delta-1}{2}}$$

for $1 \leq k \leq n$ and $\mathbf{i} \in P^n$ with $i_k = 0$. It is easy to see that if $1 \leq k \leq n-1$ and $i_k = i_{k+1} = 0$, we have $a_k(\mathbf{i}) = a_k^*(\mathbf{i}) + 1$.

3.16. Corollary. *Suppose $\mathbf{i} \in I^{n-1}$ and $\mathbf{j} = \mathbf{i} \vee 0 \in P^n$. Let $\mathbf{t} \in \mathcal{T}_{n-1}^{ud}(\mathbf{i})$ with shape (λ, f) . Then $|\mathcal{AR}_\lambda(0)| = 1$ if and only if $a_{n-1}(\mathbf{j})$ is even.*

Proof. Suppose $|\mathcal{AR}_\lambda(0)| = 1$. By Lemma 3.14, the parities of $\# \{ 1 \leq r \leq n-1 \mid i_r = 1 \} + \# \{ 1 \leq r \leq n-1 \mid i_r = -1 \}$ and $\# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} + \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \}$ are the same. As $i_1 = \frac{\delta-1}{2}$, we have $a_{n-1}^*(\mathbf{j})$ is odd by Lemma 3.15 because $|\mathcal{AR}_\lambda(0)| = 1$. This proves the only if part.

Suppose that $a_{n-1}^*(\mathbf{j})$ is odd. It forces δ to be odd. When $\delta = 1$, by the construction of young diagrams, we have

$$-1 \leq \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} \leq 1,$$

which implies that $\delta_k(\mathbf{j})$ is odd if and only if $\# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} = 0$; and when $\delta < 1$, by the construction of young diagrams, we have

$$-2 \leq \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} \leq 0,$$

which implies that $\delta_k(\mathbf{j})$ is odd if and only if $\# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} = -1$; and when $\delta > 1$, by the construction of young diagrams, we have

$$0 \leq \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = 1 \} - \# \{ \alpha \in [\lambda] \mid \text{res}(\alpha) = -1 \} \leq 2,$$

which implies that $\delta_k(\mathbf{j})$ is odd if and only if $\#\{\alpha \in [\lambda] \mid \text{res}(\alpha) = 1\} - \#\{\alpha \in [\lambda] \mid \text{res}(\alpha) = -1\} = 1$. Therefore, we have $a_{n-1}^*(\mathbf{j})$ is odd if and only if $|\mathcal{A}\mathcal{R}_\lambda(0)| = 1$ by Lemma 3.15. This proves the if part. \square

The next Corollary is easy to be verified.

3.17. Corollary. *Suppose $\mathbf{i} \in I^{n-1}$ and $\mathbf{j} = \mathbf{i} \vee 0 \in P^n$. Then $\mathbf{j} \in I^n$ if and only if $a_{n-1}(\mathbf{j})$ is even.*

Now we can extend Lemma 3.13.

3.18. Lemma. *Suppose $\mathbf{i} \in P^n$. If for any $1 \leq k \leq n$, we have either $h_k(\mathbf{i}) \in \{-2, -1\}$, or $i_k = 0$ and $a_k^*(\mathbf{i})$ is odd. Then \mathbf{i} is the residue sequence of some up-down tableau.*

Proof. We prove the Lemma by induction. When $n = 1$, the Lemma follows trivially. Assume when $n' < n$ the Lemma holds. When $n' = n$, set $\mathbf{j} = (i_1, \dots, i_{n-1})$. By induction, \mathbf{j} is the residue sequence of some up-down tableau. If $h_n(\mathbf{i}) \in \{-2, -1\}$, then $\mathbf{i} \in I^n$ by Lemma 3.13; and if $i_k = 0$ and $a_k^*(\mathbf{i})$ is odd, then $\mathbf{i} \in I^n$ by Corollary 3.17. \square

3.19. Remark. Lemma 3.18 is not sufficient to determine whether $\mathbf{i} \in P^n$ is the residue sequence of some up-down tableaux. For instance, we cannot determine whether $\mathbf{i} \in I^n$ if $i_k \neq 0$ and $h_k(\mathbf{i}) = 0$ for some $1 \leq k \leq n$. We also remark the whole argument only works when R is a field of characteristic 0. When R is a field of positive characteristic, there exists residue sequence $\mathbf{i} \in I^n$ such that $h_k(\mathbf{i}) \notin \{-2, -1, 0\}$ for some $1 \leq k \leq n$.

We close this section by giving two more results which can be used to determine whether $\mathbf{i} \cdot s_k$ is a residue sequence of some up-down tableaux by giving \mathbf{i} is a residue sequence.

3.20. Lemma. *Suppose $1 \leq k \leq n-1$ and $\mathbf{i} = (i_1, \dots, i_n)$ is a residue sequence with $i_k + i_{k+1} = 0$. Then $\mathbf{i} \cdot s_k$ is a residue sequence if and only if $h_k(\mathbf{i}) = 0$.*

Proof. Because $i_k + i_{k+1} = 0$, we have $i_k = -i_{k+1}$. Therefore, by the definition of h_k , we have $h_k(\mathbf{i} \cdot s_k) = -h_k(\mathbf{i})$. By Lemma 3.6, we have $-2 \leq h_k(\mathbf{i}) \leq 0$ as $\mathbf{i} \in I^n$, which implies $0 \leq h_k(\mathbf{i} \cdot s_k) \leq 2$. By Lemma 3.6, we have $\mathbf{i} \cdot s_k \in I^n$ only if $h_k(\mathbf{i} \cdot s_k) = 0$, which implies $h_k(\mathbf{i}) = 0$. This proves the only if part.

Assume $h_k(\mathbf{i}) = 0$. When $i_k = i_{k+1} = 0$, we have $\mathbf{i} \cdot s_k = \mathbf{i} \in I^n$; and by Lemma 3.7, we have $i_k \neq \pm \frac{1}{2}$ as $h_k(\mathbf{i}) = 0$. Hence, it only left us to prove $\mathbf{i} \cdot s_k$ is a residue sequence when $|i_k - i_{k+1}| > 1$. In this case, choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$.

Suppose $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$. When $\mathbf{t}(k) > 0, \mathbf{t}(k+1) < 0$ or $\mathbf{t}(k) < 0, \mathbf{t}(k+1) > 0$, we have that $\mathbf{t} \cdot s_k$ is an up-down tableau by Lemma 2.6; and when $\mathbf{t}(k), \mathbf{t}(k+1) > 0$ or $\mathbf{t}(k), \mathbf{t}(k+1) < 0$, because $|i_k - i_{k+1}| > 1$, the nodes $\mathbf{t}(k)$ and $\mathbf{t}(k+1)$ are not adjacent, which implies $\mathbf{t} \cdot s_k$ is an up-down tableau by Lemma 2.6. This proves $\mathbf{i} \cdot s_k$ is a residue sequence.

Suppose $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$. By Lemma 3.10, there exists a unique up-down tableau \mathbf{s} with residue sequence $\mathbf{i} \cdot s_k$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$, which implies $\mathbf{i} \cdot s_k$ is a residue sequence. This proves the if part. \square

3.21. Lemma. *Suppose $1 \leq k \leq n-1$ and $\mathbf{i} = (i_1, \dots, i_n) \in P^n$ with $|i_k - i_{k+1}| > 1$ and $i_k + i_{k+1} \neq 0$. If we have $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, then $\mathbf{t} \cdot s_k$ is an up-down tableau. In another word, we have $\mathbf{i} \cdot s_k \in I^n$ if and only if $\mathbf{i} \in I^n$.*

Proof. Suppose $\mathbf{i} \in I^n$, there exists $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. Because $i_k + i_{k+1} \neq 0$ and $|i_k - i_{k+1}| > 1$, $\mathbf{t}(k)$ and $\mathbf{t}(k+1)$ satisfy the conditions of Lemma 2.6. Hence, $\mathbf{s} = \mathbf{t} \cdot s_k$ is an up-down tableau. Because $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$, we have $\mathbf{i} \cdot s_k \in I^n$. Following the same argument, we have $\mathbf{i} \in I^n$ if $\mathbf{i} \cdot s_k \in I^n$. \square

4. Induction and restriction of $\mathcal{G}_n(\delta)$

In this section we discuss the induction and restriction properties of $\mathcal{G}_n(\delta)$. Instead of working on $\mathcal{G}_n(\delta)$ directly, first we construct a set of homogeneous elements

$$\{\psi_{\mathbf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\} \subset \mathcal{G}_n(\delta)$$

analogue to the ψ -basis of $\mathcal{K}\mathfrak{S}_n$ given in Definition 2.24 and define $R_n(\delta)$ to be the R -span of $\{\psi_{\mathbf{st}}\}$. Then we prove the induction and restriction properties of $R_n(\delta)$. In Section 5, we will prove that $R_n(\delta) = \mathcal{G}_n(\delta)$ and all the results of this section will directly apply to $\mathcal{G}_n(\delta)$.

4.1. A set of homogeneous elements of $\mathcal{G}_n(\delta)$

In this subsection we construct the set $\{\psi_{\mathbf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$, which is a set of homogeneous elements of $\mathcal{G}_n(\delta)$. Then we calculate the degree of $\psi_{\mathbf{st}}$ and show that $\deg \psi_{\mathbf{st}}$ is determined by $\deg \mathbf{s}$ and $\deg \mathbf{t}$.

Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ is an up-down tableau. Recall that we can write $\mathbf{t} = (\alpha_1, \alpha_2, \dots, \alpha_n)$. For convenience we denote $\alpha_0 = (1, 1)$. Let $\alpha_{k_1}, \alpha_{k_2}, \dots, \alpha_{k_f}$ be all negative nodes of \mathbf{t} and, for $1 \leq i \leq f$, let α_{j_i} be the first node on the left of α_{k_i} in \mathbf{t} with $\alpha_{j_i} = -\alpha_{k_i}$. So we can choose f pairs of nodes $(\alpha_{j_1}, \alpha_{k_1}), \dots, (\alpha_{j_f}, \alpha_{k_f})$ by

re-ordering these nodes such that $k_1 < k_2 < \dots < k_f$. Define the sequence to be the *remove pairs* of the up-down tableau t . Suppose $(\alpha_1, \dots, \hat{\alpha}_{j_i}, \dots, \hat{\alpha}_{k_i}, \dots, \alpha_n)$ is an up-down tableau, we say $(\alpha_{j_i}, \alpha_{k_i})$ a *removable pair* of t and i the *removable index* of t . If $(\alpha_{j_i}, \alpha_{k_i}) \neq (\alpha_0, -\alpha_0)$, we say $(\alpha_{j_i}, \alpha_{k_i})$ is a *nontrivial removable pair* of t and i the *nontrivial removable index* of t .

One can see that there exists an integer h such that for any $i \leq h$, $j_i = 2i - 1$, $k_i = 2i$, $\alpha_{j_i} = \alpha_0$ and $\alpha_{k_i} = -\alpha_0$, and $k_{h+1} \neq 2(h+1)$. We define such integer h to be the *head* of the up-down tableau t and denote $\text{head}(t) = h$.

4.1. Example. Suppose $n = 9$ and $\lambda = (1)$. We have $(\lambda, 4) \in \widehat{B}_9$. Define

$$t = (\emptyset, \square, \emptyset, \square, \square, \square, \square, \square, \square) \in \mathcal{T}_9^{ud}(\lambda).$$

We can write t as

$$t = (\alpha_1, \dots, \alpha_9) = ((1, 1), -(1, 1), (1, 1), (2, 1), (1, 2), -(2, 1), (1, 3), -(1, 3), -(1, 2)),$$

where the negative nodes are $\alpha_2, \alpha_6, \alpha_8$ and α_9 and the head $h = 1$. Moreover, we have a sequence of remove pairs $((\alpha_1, \alpha_2), (\alpha_4, \alpha_6), (\alpha_7, \alpha_8), (\alpha_5, \alpha_9))$, but not all of them are removable pairs. In more details, (α_5, α_9) is not removable and all the other pairs are removable.

4.2. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with head $h < f$. Then $h + 1$ is a removable index, i.e. $(\alpha_{j_{h+1}}, \alpha_{k_{h+1}})$ is a removable pair of t .

Proof. One can see that for any i with $h < i \leq f$, $(\alpha_{j_i}, \alpha_{k_i})$ is not removable only if there exists $h < m < i$ such that $j_i < j_m < k_m < k_i$. When $i = h + 1$, there is not such m exists, which proves the Lemma. \square

4.3. Definition. Suppose $(\lambda, f) \in \widehat{B}_n$ and $t = (\alpha_1, \dots, \alpha_n) \in \mathcal{T}_n^{ud}(\lambda)$ with head $h < f$ and remove pairs $(\alpha_{j_1}, \alpha_{k_1}), \dots, (\alpha_{j_f}, \alpha_{k_f})$. Let i be a non-trivial removable index of t and $s = (\alpha_0, -\alpha_0, \alpha_1, \dots, \hat{\alpha}_{j_i}, \dots, \hat{\alpha}_{k_i}, \dots, \alpha_n)$. We denote $s \rightarrow t$, and $\rho(s, t) = (j_i, k_i)$.

4.4. Example. Suppose t is defined as in Example 4.1. We have 2 nontrivial removable pairs of t : (α_4, α_6) and (α_7, α_8) . Define

$$\begin{aligned} s_1 &= ((1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), (1, 2), (1, 3), -(1, 3), -(1, 2)) \\ &= (\emptyset, \square, \emptyset, \square, \emptyset, \square, \square, \square, \square) \in \mathcal{T}_9^{ud}(\lambda); \\ s_2 &= ((1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), (2, 1), (1, 2), -(2, 1), -(1, 2)) \\ &= (\emptyset, \square, \emptyset, \square, \emptyset, \square, \square, \square, \square) \in \mathcal{T}_9^{ud}(\lambda). \end{aligned}$$

Hence we have $s_1 \rightarrow t$ and $s_2 \rightarrow t$, where $\rho(s_1, t) = (4, 6)$ and $\rho(s_2, t) = (7, 8)$. Notice that (α_6, α_9) is not a removable pair in t , because of the existence of (α_7, α_8) . But after we remove (α_7, α_8) , (α_6, α_9) becomes removable. As an example, set

$$\begin{aligned} s_3 &= ((1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), (2, 1), -(2, 1)) \\ &= (\emptyset, \square, \emptyset, \square, \emptyset, \square, \emptyset, \square, \square) \in \mathcal{T}_9^{ud}(\lambda). \end{aligned}$$

Then we have $s_3 \rightarrow s_2$.

The next Lemma is obvious by the definition of $s \rightarrow t$.

4.5. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$. If there exists an up-down tableau s such that $s \rightarrow t$, then $s \in \mathcal{T}_n^{ud}(\lambda)$ and $\text{head}(s) = \text{head}(t) + 1$.

Suppose $(\lambda, f) \in \widehat{B}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with head $h < f$. Because for any $s \in \mathcal{T}_n^{ud}(\lambda)$, we have $\text{head}(s) \leq f$. Hence by Lemma 4.5, there exists a finite sequence

$$t^{(m)} \rightarrow t^{(m-1)} \rightarrow \dots \rightarrow t^{(1)} \rightarrow t^{(0)} = t,$$

where $t^{(1)}, \dots, t^{(m)} \in \mathcal{T}_n^{ud}(\lambda)$ and there is no $s \in \mathcal{T}_n^{ud}(\lambda)$ such that $s \rightarrow t^{(m)}$. We define such sequence to be the *reduction sequence* of t .

4.6. Example. Suppose t is defined as in Example 4.1. Define

$$\begin{aligned} u &= ((1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1)) \\ &= (\emptyset, \square, \emptyset, \square, \emptyset, \square, \emptyset, \square, \emptyset) \in \mathcal{T}_9^{ud}(\lambda). \end{aligned}$$

Hence we have the sequence $\mathbf{t}^{(3)} \rightarrow \mathbf{t}^{(2)} \rightarrow \mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)} = \mathbf{t}$, where $\mathbf{t}^{(3)} = \mathbf{u}$,

$$\begin{aligned} \mathbf{t}^{(2)} &= ((1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), (1, 2), -(1, 2)) \\ &= (\emptyset, \square, \emptyset, \square, \emptyset, \square, \emptyset, \square, \square, \square) \in \mathcal{T}_n^{ud}(\lambda), \\ \mathbf{t}^{(1)} &= ((1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), (1, 2), (1, 3), -(1, 3), -(1, 2)) \\ &= (\emptyset, \square, \emptyset, \square, \emptyset, \square, \square, \square, \square, \square) \in \mathcal{T}_n^{ud}(\lambda), \\ \mathbf{t} = \mathbf{t}^{(0)} &= ((1, 1), -(1, 1), (1, 1), (2, 1), (1, 2), -(2, 1), (1, 3), -(1, 3), -(1, 2)) \\ &= (\emptyset, \square, \emptyset, \square, \square, \square, \square, \square, \square, \square) \in \mathcal{T}_n^{ud}(\lambda). \end{aligned}$$

In general, for $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head $h < f$, there exist more than one reduction sequence of \mathbf{t} . For example, as in Example 4.4 and Example 4.6, there exists another sequence $\mathbf{t}^{(3)} \rightarrow \mathbf{t}^{(2)} \rightarrow \mathbf{t}^{(1)} \rightarrow \mathbf{t}$ where $\mathbf{t}^{(3)} = \mathbf{u}$, $\mathbf{t}^{(2)} = \mathbf{s}_3$ and $\mathbf{t}^{(1)} = \mathbf{s}_2$.

4.7. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} = (\alpha_1, \dots, \alpha_n) \in \mathcal{T}_n^{ud}(\lambda)$ with head $h < f$ and remove pairs $(\alpha_{j_1}, \alpha_{k_1}), \dots, (\alpha_{j_f}, \alpha_{k_f})$. For any reduction sequence $\mathbf{t}^{(m)} \rightarrow \mathbf{t}^{(m-1)} \rightarrow \dots \rightarrow \mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)} = \mathbf{t}$, we have $m = f - h$.

Proof. By Lemma 4.5, $\text{head}(\mathbf{t}^{(m)}) = h + m$. If $m > f - h$, we have $\text{head}(\mathbf{t}^{(m)}) = h + m > f$. Because $\lambda \vdash n - 2f$, $\mathbf{t}^{(m)} \notin \mathcal{T}_n^{ud}(\lambda)$. Hence we have $m \leq f - h$. If $m < f - h$, then $\text{head}(\mathbf{t}^{(m)}) = h + m < f$. By Lemma 4.2 there exists $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ such that $\mathbf{s} \rightarrow \mathbf{t}^{(m)}$. Therefore we have $m = f - h$. \square

Lemma 4.7 shows that the length of the reduction sequence is determined by \mathbf{t} . Moreover, by the construction of the reduction sequence, we have $\mathbf{t}^{(f-h)} = (\beta_1, \dots, \beta_n) \in \mathcal{T}_n^{ud}(\lambda)$ where $\beta_1 = \beta_3 = \dots = \beta_{2f-1} = -\beta_2 = -\beta_4 = \dots = -\beta_{2f} = \alpha_0$, and $(\beta_{2f+1}, \beta_{2f+2}, \dots, \beta_n)$ is obtained by removing $\alpha_{j_1}, \alpha_{k_1}, \alpha_{j_2}, \alpha_{k_2}, \dots, \alpha_{j_n}, \alpha_{k_n}$ from $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$.

4.8. Example. Suppose \mathbf{t} is defined as in Example 4.1 and \mathbf{u} is in Example 4.6. We can see that by removing $(\alpha_1, \alpha_2), (\alpha_4, \alpha_6), (\alpha_7, \alpha_8), (\alpha_5, \alpha_9)$ from \mathbf{t} and add f number of $(\alpha_0, -\alpha_0)$ at the front of the resulting sequence, we have

$$((1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1), -(1, 1), (1, 1)) = \mathbf{u}.$$

Therefore $\mathbf{t}^{(f-h)}$ is uniquely determined by \mathbf{t} , and we denote $h(\mathbf{t}) = \mathbf{t}^{(f-h)}$. Moreover, by the construction of $\mathbf{t}^{(f-h)}$, one can see that if $\mathbf{t}^{(f-h)} = (\beta_1, \dots, \beta_n)$, by defining $\mathbf{s} = (\beta_{2f+1}, \beta_{2f+2}, \dots, \beta_n)$, \mathbf{s} is a tableau of shape λ .

Recall the reduction sequence of \mathbf{t} is not unique in general. A *standard reduction sequence* is a reduction sequence

$$\mathbf{t}^{(f-h)} \rightarrow \mathbf{t}^{(f-h-1)} \rightarrow \dots \rightarrow \mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)} = \mathbf{t}$$

such that for any $0 \leq i \leq f - h - 1$, $\mathbf{t}^{(i+1)}$ is obtained by removing the first non-trivial removable pairs of $\mathbf{t}^{(i)}$. In more details, suppose $\mathbf{t}^{(i)}$ has head $h + i$ and remove pairs $(\alpha_{j_1}, \alpha_{k_1}), \dots, (\alpha_{j_f}, \alpha_{k_f})$, we obtain $\mathbf{t}^{(i+1)}$ by removing $(\alpha_{j_{h+i+1}}, \alpha_{k_{h+i+1}})$, which could be done because of Lemma 4.2. As an example, the reduction sequence in Example 4.6 is a standard reduction sequence. In the rest of this paper, for any $0 \leq i \leq f - h$, $\mathbf{t}^{(i)}$ means the corresponding up-down tableau in the standard reduction sequence of \mathbf{t} .

Suppose $\mathbf{s} \rightarrow \mathbf{t}$ where \mathbf{s} has head $h + 1$ and $\rho(\mathbf{s}, \mathbf{t}) = (a, b)$. Define

$$\epsilon_{\mathbf{s} \rightarrow \mathbf{t}} := e(\mathbf{i}_{\mathbf{s}}) \epsilon_{2h+2} \epsilon_{2h+3} \dots \epsilon_a \psi_{a+1} \psi_{a+2} \dots \psi_{b-1} e(\mathbf{i}_{\mathbf{t}}) \in \mathcal{G}_n(\delta).$$

Fix $(\lambda, f) \in \widehat{B}_n$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head $h < f$ and standard reduction sequence

$$h(\mathbf{t}) = \mathbf{t}^{(f-h)} \rightarrow \mathbf{t}^{(f-h-1)} \rightarrow \dots \rightarrow \mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)} = \mathbf{t},$$

define $\epsilon_{\mathbf{t}} := \epsilon_{\mathbf{t}^{(f-h)} \rightarrow \mathbf{t}^{(f-h-1)}} \epsilon_{\mathbf{t}^{(f-h-1)} \rightarrow \mathbf{t}^{(f-h-2)}} \dots \epsilon_{\mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)}}$; and for $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f , define $\epsilon_{\mathbf{t}} := e(\mathbf{i}_{\mathbf{t}})$.

Define $\mathfrak{S}_{2f,n}$ to be the subalgebra of \mathfrak{S}_n generated by $s_{2f+1}, s_{2f+2}, \dots, s_n$. Denote $\mathbf{t}^{(\lambda, f)}$ to be the unique up-down tableau with shape (λ, f) which is maximal in dominance ordering. Suppose \mathbf{t} is an up-down tableau. Because $h(\mathbf{t})$ is uniquely determined by \mathbf{t} , we abuse the symbol and define $d(\mathbf{t}) = d(h(\mathbf{t})) \in \mathfrak{S}_{2f,n}$ to be the reduced word such that $\mathbf{t}^{\lambda} d(\mathbf{t}) = h(\mathbf{t})$. Write $d(\mathbf{t}) = s_{k_1} s_{k_2} \dots s_{k_l}$. We define

$$\psi_{\mathbf{t}} := \psi_{d(\mathbf{t})} = \psi_{k_1} \psi_{k_2} \dots \psi_{k_l}.$$

Let $\mathbf{i}_{(\lambda, f)}$ be the residue sequence of $\mathbf{t}^{(\lambda, f)}$ and define $e_{(\lambda, f)} := e(\mathbf{i}_{\lambda}) \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} e(\mathbf{i}_{\lambda})$.

4.9. Definition. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. We define $\psi_{\mathbf{s}\mathbf{t}} := \epsilon_{\mathbf{s}}^* \psi_{\mathbf{s}}^* e_{(\lambda, f)} \psi_{\mathbf{t}} \epsilon_{\mathbf{t}}$.

4.10. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. For any $\mathbf{i}, \mathbf{j} \in I^n$, we have

$$e(\mathbf{i}) \psi_{\mathbf{s}\mathbf{t}} e(\mathbf{j}) = \begin{cases} \psi_{\mathbf{s}\mathbf{t}}, & \text{if } \mathbf{i} = \mathbf{i}_{\mathbf{s}} \text{ and } \mathbf{j} = \mathbf{i}_{\mathbf{t}}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It is obvious by the definition of ψ_{st} and (3.8). \square

For an up-down tableau \mathbf{t} , $\psi_{\mathbf{t}} = \psi_{d(\mathbf{t})}$ is determined by the choice of reduced expression of $d(\mathbf{t})$. Here we prove that $\psi_{\mathbf{t}}$ is actually independent to the choice of reduced expression of $d(\mathbf{t})$.

Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. For $w = s_{r_1}s_{r_2}\dots s_{r_m} \in \mathfrak{S}_{2f,n}$, if for any $1 \leq k \leq m$, $\mathbf{t} \cdot s_{r_1}s_{r_2}\dots s_{r_k}$ is an up-down tableau, then we define $s_{r_1}s_{r_2}\dots s_{r_m}$ to be *semi-reduced correspond* to \mathbf{t} . Notice that if we have $\mathbf{s} = \mathbf{t} \cdot s_{r_1}s_{r_2}\dots s_{r_m}$ and $s_{r_1}s_{r_2}\dots s_{r_m}$ is semi-reduced correspond to \mathbf{t} , then $s_{r_m}s_{r_{m-1}}\dots s_{r_1}$ is semi-reduced correspond to \mathbf{s} .

The next Lemma can be easily verified by the definitions of semi-reduced.

4.11. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . Let $s_{r_1}s_{r_2}\dots s_{r_m} \in \mathfrak{S}_{2f,n}$ be semi-reduced correspond to \mathbf{t} , and $\mathbf{s} = \mathbf{t} \cdot s_{r_1}s_{r_2}\dots s_{r_m} \in \mathcal{T}_n^{ud}(\lambda)$ with head f and $s_{k_1}s_{k_2}\dots s_{k_l} \in \mathfrak{S}_{2f,n}$ be semi-reduced correspond to \mathbf{s} . Then $s_{r_1}s_{r_2}\dots s_{r_m}s_{k_1}s_{k_2}\dots s_{k_l} \in \mathfrak{S}_{2f,n}$ is semi-reduced correspond to \mathbf{t} .

The motivation of semi-reduced is to calculate $e(\mathbf{i}_t)\psi_w$ when w is semi-reduced.

4.12. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. If $w = s_{r_1}s_{r_2}\dots s_{r_m} \in \mathfrak{S}_{2f,n}$ is semi-reduced correspond to \mathbf{t} , then $e(\mathbf{i}_t)\psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = e(\mathbf{i}_t)\psi_w$. Equivalently, we have $\psi_{r_m}\psi_{r_{m-1}}\dots\psi_{r_1}e(\mathbf{i}_t) = \psi_{w^{-1}}e(\mathbf{i}_t)$.

Proof. It is sufficient if we can prove that

$$e(\mathbf{i}_t)\psi_k\psi_{k+1}\psi_k = e(\mathbf{i}_t)\psi_{k+1}\psi_k\psi_{k+1}, \quad e(\mathbf{i}_t)\psi_k^2 = e(\mathbf{i}_t), \quad e(\mathbf{i}_t)\psi_k\psi_r = e(\mathbf{i}_t)\psi_r\psi_k,$$

under the assumptions of the Lemma, which can be verified directly by checking the relations of $\mathcal{G}_n(\delta)$ and the construction of up-down tableaux. \square

4.13. Example. Suppose $\mathbf{t} = (\emptyset, \square, \square\square)$. Then $\mathbf{t} \cdot s_1^2 = \mathbf{t}$ but $\mathbf{t} \cdot s_1$ is not an up-down tableau. So s_1^2 is not semi-reduced correspond to \mathbf{t} .

Suppose $\mathbf{t} = (\emptyset, \square, \square\square, \square\square)$. Then $\mathbf{t} \cdot s_2^2 = \mathbf{t}$, and $\mathbf{t} \cdot s_2 = (\emptyset, \square, \square, \square\square)$ is an up-down tableau. Then s_2^2 is semi-reduced correspond to \mathbf{t} . Moreover, choose $\delta = 1$. Then the residue sequence $\mathbf{i}_t = (0, 1, 2, -1)$. Hence $e(\mathbf{i}_t)\psi_2^2 = e(\mathbf{i}_t)$.

The next Corollary is implied by Lemma 4.12.

4.14. Corollary. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. If $w \in \mathfrak{S}_n$ is semi-reduced correspond to \mathbf{t} , then we have $e(\mathbf{i}_t)\psi_w\psi_{w^{-1}} = e(\mathbf{i}_t)$.

Proof. Let $\mathbf{s} = \mathbf{t} \cdot w \in \mathcal{T}_n^{ud}(\lambda)$. By the definition of semi-reduced, w^{-1} is semi-reduced correspond to \mathbf{s} . Consider ww^{-1} as a word in \mathfrak{S}_n . By Lemma 4.11, ww^{-1} is semi-reduced correspond to \mathbf{t} . Therefore by Lemma 4.12, we have $e(\mathbf{i}_t)\psi_w\psi_{w^{-1}} = e(\mathbf{i}_t)$. \square

Generally, reduced does not imply semi-reduced. The next Lemma gives a special case when reduced implies semi-reduced. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . Recall $d(\mathbf{t}) \in \mathfrak{S}_n$ such that $\mathbf{t}^{(\lambda, f)} \cdot d(\mathbf{t}) = \mathbf{t}$.

4.15. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . Then $d(\mathbf{t}) = s_{r_1}s_{r_2}\dots s_{r_m} \in \mathfrak{S}_{2f,n}$ is reduced implies $d(\mathbf{t})$ is semi-reduced correspond to $\mathbf{t}^{(\lambda, f)}$.

Proof. It is straightforward by the definition of $d(\mathbf{t})$. \square

Suppose \mathbf{t} is an up-down tableau. Lemma 4.12 and Lemma 4.15 imply $e(\mathbf{i}_{(\lambda, f)})\psi_{\mathbf{t}}$ is independent to the choice of $d(\mathbf{t})$.

By the definition of ψ_{st} , one can see that ψ_{st} 's are homogeneous. Next we calculate the degree of ψ_{st} 's. Let $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. First we consider the simplest case, which is $head(\mathbf{s}) = head(\mathbf{t}) = f$.

4.16. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . Then $\deg \mathbf{t} = \frac{1}{2} \deg e_{(\lambda, f)} + \deg \psi_{d(\mathbf{t})}e(\mathbf{i}_t) = \frac{1}{2} \deg e_{(\lambda, f)}$.

Proof. Suppose $\mathbf{u} \in \mathcal{T}_n^{ud}(\lambda)$ with head f and residue sequence $\mathbf{i}_u = (i_1, \dots, i_n)$. By the construction of \mathbf{u} , we have $u(k) > 0$ for any $2f + 1 \leq k \leq n$. If $\mathbf{s} = \mathbf{u} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$ for some $s_k \in \mathfrak{S}_{2f,n}$, then $s(k) = u(k + 1) > 0$ and $s(k + 1) = u(k) > 0$. Therefore, we have

$$\deg \mathbf{s} - \deg \mathbf{u} = \deg(\mathbf{s}|_{k-1} \Rightarrow \mathbf{s}|_k) + \deg(\mathbf{s}|_k \Rightarrow \mathbf{s}|_{k+1}) - \deg(\mathbf{u}|_{k-1} \Rightarrow \mathbf{u}|_k) - \deg(\mathbf{u}|_k \Rightarrow \mathbf{u}|_{k+1}) = 0,$$

by Lemma 4.24. Moreover, as the nodes $u(k)$ and $u(k + 1)$ are not adjacent by Lemma 2.6, we have $|i_k - i_{k+1}| > 1$. Therefore, we have $\deg \mathbf{s} - \deg \mathbf{u} = 0 = \deg \psi_k e(\mathbf{i}_t)$. Hence, as $\mathbf{t}^{(\lambda, f)} \in \mathcal{T}_n^{ud}(\lambda)$ with head f and $d(\mathbf{t}) \in \mathfrak{S}_{2f,n}$, we have $\deg \mathbf{t} - \deg \mathbf{t}^{(\lambda, f)} = 0 = \deg \psi_{d(\mathbf{t})}e(\mathbf{i}_t)$.

It suffices to prove that when $t = t^{(\lambda, f)}$ we have $\deg t = \frac{1}{2} \deg e_{(\lambda, f)}$. By direct calculation, we have $\deg t^{(\lambda, f)} = 0$ if $\delta \neq 0$ and $\deg t^{(\lambda, f)} = f$ if $\delta = 0$. Also as

$$\deg e_{(\lambda, f)} = \begin{cases} 2f, & \text{if } \delta = 0, \\ 0, & \text{if } \delta \neq 0, \end{cases}$$

we have $\deg t^{(\lambda, f)} = \frac{1}{2} \deg e_{(\lambda, f)}$, which completes the proof. \square

4.17. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $s, t \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head}(s) = \text{head}(t) = f$. Then we have $\deg \psi_{st} = \deg s + \deg t$.

Proof. By the definition of ψ_{st} 's, we have $\psi_{st} = \psi_s^* e_{(\lambda, f)} \psi_t$. Hence, by Lemma 4.16, we have

$$\begin{aligned} \deg \psi_{st} &= \deg e(\mathbf{i}_s) \psi_s^* + \deg e_{(\lambda, f)} + \deg \psi_t e(\mathbf{i}_t) \\ &= \frac{1}{2} \deg e_{(\lambda, f)} + \deg \psi_s e(\mathbf{i}_s) + \frac{1}{2} \deg e_{(\lambda, f)} + \deg \psi_s e(\mathbf{i}_t) \\ &= \deg s + \deg t, \end{aligned}$$

which proves the Lemma. \square

Next we extend Lemma 4.16 to arbitrary $t \in \mathcal{T}_n^{ud}(\lambda)$ to show that

$$\deg \psi_{st} = \deg s + \deg t. \quad (4.1)$$

Here we give some examples with the up-down tableau t in Example 2.9 and 2.10, and compare the values of $\frac{1}{2} \deg e_{(\lambda, f)} + \deg e(\mathbf{i}_{(\lambda, f)}) \psi_t \epsilon_1$ with $\deg t$ calculated in Example 2.9 and 2.10.

4.18. Example. Let $n = 6$, $\lambda = (1, 1)$, $\delta = 1$ and $t = (\emptyset, \square, \square, \square, \square, \square) \in \mathcal{T}_n^{ud}(\lambda)$. We have the standard reduction sequence $h(t) = t^{(2)} \rightarrow t^{(1)} \rightarrow t^{(0)} = t$ where

$$\begin{aligned} t^{(1)} &= (\emptyset, \square, \emptyset, \square, \square, \square), \\ t^{(2)} &= (\emptyset, \square, \emptyset, \square, \emptyset, \square), \end{aligned}$$

and $\rho(t^{(2)}, t^{(1)}) = (4, 6)$, $\rho(t^{(1)}, t^{(0)}) = (4, 5)$. Henceforth we have

$$e_{(\lambda, f)} \psi_t \epsilon_1 = e(0, 0, 0, 0, 0, -1) \epsilon_1 \epsilon_3 e(0, 0, 0, 0, 0, -1) \epsilon_4 \psi_5 e(0, 0, 0, 1, -1, -1) \epsilon_2 \epsilon_3 \epsilon_3 e(0, 1, -1, 0, 0, -1).$$

By the direct calculations, the degree of elements $\frac{1}{2} \deg e_{(\lambda, f)} + \deg e(\mathbf{i}_{(\lambda, f)}) \psi_t \epsilon_1$ is

$$\frac{1}{2} \deg e_{(\lambda, f)} + \deg e(\mathbf{i}_{(\lambda, f)}) \psi_t \epsilon_1 = \frac{1}{2} \times 0 + 1 - 2 + 1 - 2 + 1 = -1,$$

which is the same as $\deg t$.

4.19. Example. Let $n = 6$, $\lambda = (1, 1)$, $\delta = 0$ and $t = (\emptyset, \square, \square, \square, \square, \square) \in \mathcal{T}_n^{ud}(\lambda)$. We have the standard reduction sequence $h(t) = t^{(2)} \rightarrow t^{(1)} \rightarrow t^{(0)} = t$ where

$$\begin{aligned} t^{(1)} &= (\emptyset, \square, \emptyset, \square, \square, \square), \\ t^{(2)} &= (\emptyset, \square, \emptyset, \square, \emptyset, \square), \end{aligned}$$

and $\rho(t^{(2)}, t^{(1)}) = (4, 6)$, $\rho(t^{(1)}, t^{(0)}) = (4, 5)$. Henceforth we have

$$e_{(\lambda, f)} \psi_t \epsilon_1 = e(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}) \epsilon_1 \epsilon_3 e(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}) \epsilon_4 \psi_5 e(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}) \epsilon_2 \epsilon_3 \epsilon_3 e(-\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

By the direct calculations, the degree of elements $\frac{1}{2} \deg e_{(\lambda, f)} + \deg e(\mathbf{i}_{(\lambda, f)}) \psi_t \epsilon_1$ is

$$\frac{1}{2} \deg e_{(\lambda, f)} + \deg e(\mathbf{i}_{(\lambda, f)}) \psi_t \epsilon_1 = \frac{1}{2} \times 4 - 2 + 1 - 2 + 2 - 1 = 0,$$

which is the same as $\deg t$.

Suppose $(\lambda, f) \in \widehat{B}_n$ and fix $t \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head } h \leq f$. Write $\epsilon_1 = e(\mathbf{i}_{h(t)}) g_1 g_2 \dots g_m e(\mathbf{i}_t)$ where $g_i \in \{\psi_k, \epsilon_k \mid 1 \leq k \leq n-1\}$ for $1 \leq i \leq m$. We will extend Lemma 4.16 by showing $\deg t - \deg h(t) = \deg \epsilon_1$ using induction on m . The base case, $m = 0$, is proved by Lemma 4.16. In order to complete the induction process, we need to show that there exists $s \in \mathcal{T}_n^{ud}(\lambda)$ such that $\epsilon_s e(\mathbf{i}_s) g_m e(\mathbf{i}_t) = \epsilon_1$.

Let $t^{(f-h)} \rightarrow t^{(f-h-1)} \rightarrow \dots \rightarrow t^{(0)} = t$ be the standard reduction sequence of t . Fix i with $0 \leq i \leq f-h-1$ and write $u = t^{(i+1)}$ and $v = t^{(i)}$. By the definition we have $u \rightarrow v$.

4.20. Lemma. Suppose $u, v \in \mathcal{T}_n^{ud}(\lambda)$ are defined as above. If $\rho(u, v) = (a, b)$, we have $v(\ell) > 0$ for any ℓ with $2(i+h) < \ell < b$. Moreover, the node $v(\ell)$ is not adjacent to $v(a)$.

Proof. Suppose $v = (\alpha_1, \dots, \alpha_n)$ has remove pairs $(\alpha_{j_1}, \alpha_{k_1}), \dots, (\alpha_{j_f}, \alpha_{k_f})$. By the definition of standard reduction sequence, we have $k_{i+h} = 2(i+h)$ and $k_{i+h+1} = b$. Therefore the Lemma follows because $\alpha_{k_1}, \dots, \alpha_{k_f}$ are all the negative nodes of $\alpha_1, \dots, \alpha_n$ and $k_1 < k_2 < \dots < k_f$.

Now assume $v(\ell)$ is adjacent to $v(a)$. Then $v(\ell)$ is either below or on the right of $v(a)$. Hence $v(a) \notin \mathcal{R}(v_{b-1})$, which contradicts to $\rho(u, v) = (a, b)$. \square

4.21. Lemma. Suppose $u, v \in \mathcal{T}_n^{ud}(\lambda)$ are defined as above and write $\epsilon_{u \rightarrow v} = e(\mathbf{i}_u)g_1 \dots g_m e(\mathbf{i}_v)$ where $g_i \in \{\psi_k, \epsilon_k \mid 1 \leq k \leq n-1\}$ for $1 \leq i \leq m$. Then there exists $s \in \mathcal{T}_n^{ud}(\lambda)$ with $u \rightarrow s$ such that $\epsilon_{u \rightarrow s} e(\mathbf{i}_s)g_m e(\mathbf{i}_v) = \epsilon_{u \rightarrow v}$. Moreover, the following results hold:

- (1) If $g_m = \psi_k$ for some $1 \leq k \leq n-1$, then $s(k) < 0$, $s(k+1) > 0$ and $s = v \cdot s_k$.
- (2) If $g_m = \epsilon_k$ for some $1 \leq k \leq n-1$, then $s(k) = -s(k+1) < 0$ and $v(k) = -v(k+1) > 0$.

Proof. Suppose $v = (\alpha_1, \dots, \alpha_n)$ and $\rho(u, v) = (a, b)$. We have $\alpha_a = -\alpha_b > 0$.

(1). When $g_m = \psi_k$, we have $a < b-1 = k$ by the definition of $\epsilon_{u \rightarrow v}$. Hence $\alpha_k \neq \alpha_a$ and by Lemma 4.20, $\alpha_k > 0$. Therefore, we have $v(k) = \alpha_k > 0$, $v(k+1) = \alpha_b = -\alpha_a < 0$ and $v(k) + v(k+1) \neq 0$. By Lemma 2.6, $v \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$. Let $s = v \cdot s_k$ and we have $\epsilon_{u \rightarrow s} = e(\mathbf{i}_u)g_1 \dots g_{m-1} e(\mathbf{i}_s)$ by the definition of $\epsilon_{u \rightarrow s}$ and $s(k) = \alpha_{k+1} < 0$, $s(k+1) = \alpha_k > 0$.

(2). When $g_m = \epsilon_k$, let $\text{head}(u) = \ell$. By the definition of $\epsilon_{u \rightarrow v}$ we have $a = b-1 = k$ and $\epsilon_{u \rightarrow v} = e(\mathbf{i}_u)\epsilon_{2\ell}\epsilon_{2\ell+1} \dots \epsilon_k e(\mathbf{i}_v)$. Hence we have $v(k) = -v(k+1) > 0$. Let $s = (\alpha_1, \dots, \alpha_{k-1}, -\alpha_{k-1}, \alpha_{k-1}, \alpha_{k+2}, \dots, \alpha_n)$, by the construction we have $\epsilon_{u \rightarrow s} = e(\mathbf{i}_u)\epsilon_{2\ell}\epsilon_{2\ell+1} \dots \epsilon_{k-1} e(\mathbf{i}_s) = e(\mathbf{i}_u)g_1 \dots g_{m-1} e(\mathbf{i}_s)$. By Lemma 4.20 we have $\alpha_{k-1} > 0$. Hence we have $s(k) = -s(k+1) < 0$. \square

The next Corollary is a direct result of Lemma 4.21.

4.22. Corollary. Suppose $(\lambda, f) \in \widehat{B}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$. Write $\epsilon_t = e(\mathbf{i}_{h(t)})g_1 g_2 \dots g_m e(\mathbf{i}_t)$ where $g_i \in \{\psi_k, \epsilon_k \mid 1 \leq r \leq n-1\}$ for $1 \leq i \leq m$. Then there exists $s \in \mathcal{T}_n^{ud}(\lambda)$ such that $\epsilon_s e(\mathbf{i}_s)g_m e(\mathbf{i}_t) = \epsilon_t$. Moreover, the following results hold:

- (1) If $g_m = \psi_k$ for some $1 \leq k \leq n-1$, then $s(k-1) < 0$, $s(k) > 0$ and $s = t \cdot s_k$.
- (2) If $g_m = \epsilon_k$ for some $1 \leq k \leq n-1$, then $s(k) = -s(k+1) < 0$ and $t(k) = -t(k+1) > 0$.

Corollary 4.22 shows that there exists $s \in \mathcal{T}_n^{ud}(\lambda)$ such that $\epsilon_s e(\mathbf{i}_s)g_m e(\mathbf{i}_t) = \epsilon_t$ for $g_m \in \{\psi_k, \epsilon_k \mid 1 \leq k \leq n-1\}$. In order to complete the induction process, we want to prove that

$$\deg t - \deg s = \deg \epsilon_t - \deg \epsilon_s = \deg e(\mathbf{i}_s)g_m e(\mathbf{i}_t). \quad (4.2)$$

Suppose $g_m \in \{\epsilon_k, \psi_k\}$ for some $1 \leq k \leq n-1$. By Corollary 4.22, we have $t(r) = s(r)$ for any $r \neq k, k+1$. Therefore, we can re-write (4.2) as

$$\deg(t|_{k-1} \Rightarrow t|_k) + \deg(t|_k \Rightarrow t|_{k+1}) - \deg(s|_{k-1} \Rightarrow s|_k) - \deg(s|_k \Rightarrow s|_{k+1}) = \deg e(\mathbf{i}_s)g_m e(\mathbf{i}_t). \quad (4.3)$$

First we prove (4.3) when $g_m = \epsilon_k$ for some $1 \leq k \leq n-1$. Notice that $\deg e(\mathbf{i}_s)\epsilon_k e(\mathbf{i}_t) = \deg_k(\mathbf{i}_s) + \deg_k(\mathbf{i}_t)$. The following results connect $\deg(t|_{k-1} \Rightarrow t|_k)$ and $\deg_k(\mathbf{i}_t)$.

4.23. Lemma. Suppose t is an up-down tableau of size n and $1 \leq k \leq n$. If $t_k \subset t_{k-1}$, we have

$$\deg(t|_{k-1} \Rightarrow t|_k) = -\deg_k(\mathbf{i}_t).$$

Proof. Denote $\lambda = t_{k-1}$ and $\mu = t_k$. As $\mu \subset \lambda$, there exists a positive node α such that $\mu = \lambda \setminus \{\alpha\}$. Moreover, if we write $\mathbf{i}_t = (i_1, \dots, i_n)$, we have $i_k = -\text{res}(\alpha) = \text{res}_\lambda(\alpha)$. Because $\mathbf{i}_t \in I^n = I_{k,+}^n \sqcup I_{k,-}^n \sqcup I_{k,0}^n$, we prove the Lemma by considering the following cases:

Case 1: $\mathbf{i}_t \in I_{k,0}^n$ and $i_k = -\text{res}(\alpha) = \text{res}_\lambda(\alpha) \neq -\frac{1}{2}$.

When $i_k \neq 0, \frac{1}{2}$, we have $h_k(\mathbf{i}_t) = -1$. By (3.6) we have $\mathcal{AR}_\lambda(i_k) = \{\alpha\} \subset \mathcal{R}(\lambda)$ and $\mathcal{AR}_\lambda(-i_k) = \emptyset$. As $\mu = \lambda \setminus \{\alpha\}$ and $\text{res}(\alpha) \neq \pm\frac{1}{2}$, by the construction we have $\widehat{\mathcal{A}}_t(k) = \widehat{\mathcal{R}}_t(k) = \emptyset$. Therefore

$$\deg(t|_{k-1} \Rightarrow t|_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = 0 = -\deg_k(\mathbf{i}_t).$$

When $i_k = 0$, we have $h_k(\mathbf{i}_t) = 0$. By Lemma 3.8 we have $\mathcal{AR}_\lambda(i_k) = \{\alpha\} \subset \mathcal{R}(\lambda)$. As $\mu = \lambda \setminus \{\alpha\}$, by the construction we have $\widehat{\mathcal{A}}_t(k) = \widehat{\mathcal{R}}_t(k) = \emptyset$. Therefore

$$\deg(t|_{k-1} \Rightarrow t|_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = 0 = -\deg_k(\mathbf{i}_t).$$

When $i_k = \frac{1}{2}$, we have $\text{res}(\alpha) = -\frac{1}{2}$ and $h_k(\mathbf{i}_t) = -1$. Write $\alpha = (i, j) \in [\lambda]$. By (3.6) we have $\mathcal{AR}_\lambda(i_k) = \{\alpha\} \subset \mathcal{R}(\lambda)$. Hence $(i, j+1) \notin \mathcal{A}(\lambda)$, which implies $(i-1, j+1) \notin [\lambda]$. Hence we have $(i, j), (i-1, j+1) \notin [\mu]$. Therefore $\gamma = (i-1, j) \in \mathcal{R}(\mu)$, and $\text{res}(\gamma) = \frac{1}{2} = i_k$. Hence $\widehat{\mathcal{A}}_t(k) = \emptyset$ and $\widehat{\mathcal{R}}_t(k) = \{\gamma\}$, which yields

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = 0 = -\deg_k(\mathbf{i}_t).$$

Case 2: $\mathbf{i}_t \in I_{k,0}^n$ and $i_k = -\text{res}(\alpha) = \text{res}_\lambda(\alpha) = -\frac{1}{2}$.

As $\mathbf{i}_t \in I_{k,0}^n$ and $i_k = -\frac{1}{2}$, we have $h_k(\mathbf{i}_t) = -2$. Write $\alpha = (i, j)$ and $\gamma = (i+1, j)$. By Lemma 3.8 we have $\mathcal{AR}_\lambda(i_k) = \{\alpha, \gamma\}$ where $\alpha \in \mathcal{R}(\lambda)$ and $\gamma \in \mathcal{A}(\lambda)$. As $\alpha \notin [\mu]$, one can see that $\gamma \notin \mathcal{AR}(\mu)$. As $\gamma \in \mathcal{A}(\lambda)$, we have $(i+1, j-1) \in [\lambda] \cap [\mu]$. Because $\alpha \notin [\mu]$, we have $(i, j-1) \notin \mathcal{AR}(\mu)$. Therefore we have $\widehat{\mathcal{A}}_t(k) = \widehat{\mathcal{R}}_t(k) = \emptyset$, which yields

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = 0 = -\deg_k(\mathbf{i}_t).$$

Case 3: $\mathbf{i}_t \in I_{k,-}^n$ and $i_k = -\text{res}(\alpha) = \text{res}_\lambda(\alpha) \neq \frac{1}{2}$.

As $\mathbf{i}_t \in I_{k,-}^n$, we have $h_k(\mathbf{i}_t) = -2$. By Lemma 3.8 we have $\mathcal{AR}_\lambda(i_k) = \{\alpha, \gamma\}$ where $\gamma \in \mathcal{A}(\lambda)$ with $\text{res}(\gamma) = i_k$, and $\mathcal{AR}_\lambda(-i_k) = \emptyset$. Because $i_k \neq \pm\frac{1}{2}$, we have $\gamma \in \mathcal{A}(\mu)$, which implies $\widehat{\mathcal{A}}_t(k) = \{\gamma\}$ and $\widehat{\mathcal{R}}_t(k) = \emptyset$. Therefore

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = 1 = -\deg_k(\mathbf{i}_t).$$

Case 4: $\mathbf{i}_t \in I_{k,-}^n$ and $i_k = -\text{res}(\alpha) = \text{res}_\lambda(\alpha) = \frac{1}{2}$.

As $\mathbf{i}_t \in I_{k,-}^n$, we have $h_k(\mathbf{i}_t) = -2$. Write $\alpha = (i, j)$ and $\gamma = (i, j+1)$. By Lemma 3.8 we have $\mathcal{AR}_\lambda(i_k) = \{\alpha, \gamma\}$ where $\alpha \in \mathcal{R}(\lambda)$ and $\gamma \in \mathcal{A}(\lambda)$. As $\alpha \notin [\mu]$, one can see that $\gamma \notin \mathcal{AR}(\mu)$. As $\gamma \in \mathcal{A}(\lambda)$, we have $(i-1, j+1) \in [\lambda] \cap [\mu]$. Because $\alpha \notin [\mu]$, we have $(i-1, j) \notin \mathcal{AR}(\mu)$. Therefore we have $\widehat{\mathcal{A}}_t(k) = \widehat{\mathcal{R}}_t(k) = \emptyset$, which implies

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = 1 = -\deg_k(\mathbf{i}_t).$$

Case 5: $\mathbf{i}_t \in I_{k,+}^n$ and $i_k = -\text{res}(\alpha) = \text{res}_\lambda(\alpha) \neq -\frac{1}{2}$.

As $\mathbf{i}_t \in I_{k,+}^n$ and $i_k \neq -\frac{1}{2}$, we have $h_k(\mathbf{i}_t) = 0$. Hence by Lemma 3.8 we have $\mathcal{AR}_\lambda(-i_k) = \{\gamma\}$ and $\mathcal{AR}_\lambda(i_k) = \{\alpha\}$, where $\gamma \in \mathcal{R}(\lambda)$ and $\text{res}(\gamma) = i_k = -\text{res}(\alpha)$. As $i_k \neq \pm\frac{1}{2}$, we have $\gamma \in \mathcal{R}(\mu)$, which implies $\widehat{\mathcal{A}}_t(k) = \emptyset$ and $\widehat{\mathcal{R}}_t(k) = \{\gamma\}$. Therefore we have

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = -1 = -\deg_k(\mathbf{i}_t).$$

Case 6: $\mathbf{i}_t \in I_{k,+}^n$ and $i_k = -\text{res}(\alpha) = \text{res}_\lambda(\alpha) = -\frac{1}{2}$.

As $\mathbf{i}_t \in I_{k,+}^n$ and $i_k = \frac{1}{2}$, we have $h_k(\mathbf{i}_t) = -1$. Write $\alpha = (i, j) \in [\lambda]$. By (3.6) we have $\mathcal{AR}_\lambda(i_k) = \{\alpha\} \subset \mathcal{R}(\lambda)$ and $\mathcal{AR}_\lambda(-i_k) = \emptyset$. Hence $(i+1, j) \notin \mathcal{A}(\lambda)$, which implies $(i+1, j-1) \notin [\lambda]$. Hence we have $(i, j), (i+1, j-1) \notin [\mu]$. Therefore $\gamma = (i, j-1) \in \mathcal{R}(\mu)$, and $\text{res}(\gamma) = -\frac{1}{2} = i_k$. Hence $\widehat{\mathcal{A}}_t(k) = \emptyset$ and $\widehat{\mathcal{R}}_t(k) = \{\gamma\}$, which implies

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) = |\widehat{\mathcal{A}}_t(k)| - |\widehat{\mathcal{R}}_t(k)| + \delta_{\text{res}(\alpha), -\frac{1}{2}} = -1 = -\deg_k(\mathbf{i}_t). \quad \square$$

4.24. Lemma. Suppose \mathbf{t} is an up-down tableau of size n and $0 \leq k < n$. If $\mathbf{t}_k \subset \mathbf{t}_{k+1}$, we have

$$\deg(\mathbf{t}_k \Rightarrow \mathbf{t}_{k+1}) = 0.$$

Proof. It is easy to verify because $|\mathcal{A}_t(k)| = |\mathcal{R}_t(k)| = 0$. \square

4.25. Lemma. Suppose $\mathbf{i} \in I^n$ and $1 \leq k \leq n-2$. If $i_k = -i_{k+1} = i_{k+2}$, we have $\deg_k(\mathbf{i}) = -\deg_{k+1}(\mathbf{i})$.

Proof. Because $i_k = -i_{k+1}$, by applying (3.2) to the definitions of $I_{k,0}^n$, $I_{k,-}^n$ and $I_{k,+}^n$, we can see that $\mathbf{i} \in I_{k,0}^n$ implies $\mathbf{i} \in I_{k+1,0}^n$, $\mathbf{i} \in I_{k,-}^n$ implies $\mathbf{i} \in I_{k+1,+}^n$ and $\mathbf{i} \in I_{k,+}^n$ implies $\mathbf{i} \in I_{k+1,-}^n$. Hence we have $\deg_k(\mathbf{i}) = -\deg_{k+1}(\mathbf{i})$ by the definition of $\deg_k(\mathbf{i})$. \square

4.26. Corollary. Suppose \mathbf{t} is an up-down tableau of size n and $1 \leq k < n$. If $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$, we have

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) + \deg(\mathbf{t}_k \Rightarrow \mathbf{t}_{k+1}) = \begin{cases} \deg_k(\mathbf{i}_t), & \text{if } \mathbf{t}_{k-1} \subset \mathbf{t}_k, \\ -\deg_k(\mathbf{i}_t), & \text{if } \mathbf{t}_k \subset \mathbf{t}_{k-1}. \end{cases}$$

Proof. When $\mathbf{t}_k \subset \mathbf{t}_{k-1}$, by Lemma 4.23 and Lemma 4.24, we have

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) + \deg(\mathbf{t}_k \Rightarrow \mathbf{t}_{k+1}) = -\deg_k(\mathbf{i}_t).$$

When $\mathbf{t}_{k-1} \subset \mathbf{t}_k$, because $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$, we have $\mathbf{t}_{k+1} = \mathbf{t}_{k-1} \subset \mathbf{t}_k$. By Lemma 4.23 and Lemma 4.24, we have

$$\deg(\mathbf{t}_{k-1} \Rightarrow \mathbf{t}_k) + \deg(\mathbf{t}_k \Rightarrow \mathbf{t}_{k+1}) = -\deg_{k+1}(\mathbf{i}_t),$$

and by Lemma 4.25 we have $\deg_k(\mathbf{i}_t) = -\deg_{k+1}(\mathbf{i}_t)$, which completes the proof. \square

Now we are ready to prove (4.3) when $g_m = \epsilon_k$.

4.27. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ such that $\epsilon_k = \epsilon_s e(\mathbf{i}_s) \epsilon_k e(\mathbf{i}_t)$ for $1 \leq k \leq n-1$. Then the equality (4.3) holds.

Proof. By Corollary 4.22, we have $s(k) = -s(k+1) < 0$ and $t(k) = -t(k+1) > 0$. Hence by Corollary 4.26, we have

$$\begin{aligned} \deg(s|_{k-1} \Rightarrow s|_k) + \deg(s|_k \Rightarrow s|_{k+1}) &= -\deg_k(\mathbf{i}_s), \\ \deg(t|_{k-1} \Rightarrow t|_k) + \deg(t|_k \Rightarrow t|_{k+1}) &= \deg_k(\mathbf{i}_t). \end{aligned}$$

As $s(r) = t(r)$ for any $r \neq k, k+1$, we have

$$\begin{aligned} &\deg(t|_{k-1} \Rightarrow t|_k) + \deg(t|_k \Rightarrow t|_{k+1}) - \deg(s|_{k-1} \Rightarrow s|_k) - \deg(s|_k \Rightarrow s|_{k+1}) \\ &= \deg_k(\mathbf{i}_s) + \deg_k(\mathbf{i}_t) = \deg e(\mathbf{i}_s) \epsilon_k e(\mathbf{i}_t), \end{aligned}$$

which proves the Lemma. \square

Then we prove (4.3) when $g_m = \psi_k$.

4.28. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ such that $\epsilon_k = \epsilon_s e(\mathbf{i}_s) \psi_k e(\mathbf{i}_t)$ for $1 \leq k \leq n-1$. Then the equality (4.3) holds.

Proof. By Corollary 4.22, we have $s(k-1) < 0$, $s(k) > 0$ and $t = s \cdot s_k$. By Lemma 4.23 and Lemma 4.24, we have

$$\begin{aligned} &\deg(t|_{k-1} \Rightarrow t|_k) + \deg(t|_k \Rightarrow t|_{k+1}) - \deg(s|_{k-1} \Rightarrow s|_k) - \deg(s|_k \Rightarrow s|_{k+1}) \\ &= \deg(t|_k \Rightarrow t|_{k+1}) - \deg(s|_{k-1} \Rightarrow s|_k) = \deg_k(\mathbf{i}_s) - \deg_{k+1}(\mathbf{i}_t). \end{aligned} \quad (4.4)$$

Let $\mathbf{i}_s = (i_1, \dots, i_n)$. Notice $\mathbf{i}_t = \mathbf{i}_s \cdot s_k$. If $|i_k - i_{k+1}| > 1$, we have $h_k(\mathbf{i}_s) = h_{k+1}(\mathbf{i}_t)$, which implies that $\mathbf{i}_s \in I_{k,a}^n$ if and only if $\mathbf{i}_t \in I_{k+1,a}^n$ for $a \in \{+, -, 0\}$. Hence we have $\deg_k(\mathbf{i}_s) - \deg_{k+1}(\mathbf{i}_t) = 0$. By (4.4), the Lemma holds.

If $i_k = i_{k+1} \pm 1$, first we exclude some of the cases: when $i_k = 0$ and $s(k) < 0$, we always $s(k+1) < 0$ when $i_k = i_{k+1} \pm 1$, which does not satisfy the condition of the Lemma; and when $i_k = \pm \frac{1}{2}$ and $i_{k+1} = -i_k$, we always have $s(k) + s(k+1) = 0$ or $s(k+1) < 0$, which does not satisfy the condition of the Lemma; and when $h_k(\mathbf{i}_s) = -2$, by the construction of \mathbf{s} there exists no up-down tableau \mathbf{s} satisfying the conditions of the Lemma. By excluding these cases, we have $h_{k+1}(\mathbf{i}_t) = h_k(\mathbf{i}_s) - 1$. By direct calculations, we have $\mathbf{i}_s \in I_{k,0}^n$ if and only if $\mathbf{i}_t \in I_{k+1,-}^n$, and $\mathbf{i}_s \in I_{k,+}^n$ if and only if $\mathbf{i}_t \in I_{k+1,0}^n$. Hence we have $\deg_k(\mathbf{i}_s) - \deg_{k+1}(\mathbf{i}_t) = 1$. By (4.4), the Lemma holds.

If $i_k = i_{k+1}$, by the construction of \mathbf{s} we have $h_k(\mathbf{i}_s) = -2$. First we exclude some of the cases: when $i_k = i_{k+1} = \pm \frac{1}{2}$, we have $h_{k+1}(\mathbf{i}_s) = h_k(\mathbf{i}_s) + 3 = 1$, which implies $\mathbf{i}_s \notin I^n$ by Lemma 3.6; and when $i_k = 0$, we have $h_k(\mathbf{i}_s) = 0$, which contradicts that $h_k(\mathbf{i}_s) = -2$. Therefore we have $i_k = i_{k+1} \neq 0, \pm \frac{1}{2}$, which yields $h_{k+1}(\mathbf{i}_t) = h_k(\mathbf{i}_s) + 2 = 0$ by (3.1). Hence, we have $\mathbf{i}_s \in I_{k,-}^n$, $\mathbf{i}_t \in I_{k+1,+}^n$, and $\deg_k(\mathbf{i}_s) - \deg_{k+1}(\mathbf{i}_t) = -2$. By (4.4), the Lemma holds.

Therefore, we have considered all the cases, which completes the proof. \square

Now we are able to give a proper proof for the equality (4.1).

4.29. Proposition. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. We have $\deg \psi_{st} = \deg \mathbf{s} + \deg \mathbf{t}$.

Proof. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$, let $head(\mathbf{t}) = h$. By the definition, we can write $\epsilon_k = e(\mathbf{i}_{h(\mathbf{t})}) g_1 \dots g_m e(\mathbf{i}_t)$ where $g_i \in \{\psi_k, \epsilon_k \mid 1 \leq k \leq n-1\}$ for $1 \leq i \leq m$. First we prove that

$$\deg \mathbf{t} - \deg h(\mathbf{t}) = \deg \epsilon_k. \quad (4.5)$$

We apply induction here. For base case, it is obvious that when $m = 0$ the equality (4.2) holds, because when $m = 0$, $\mathbf{t} = h(\mathbf{t})$ and $\epsilon_k = e(\mathbf{i}_t)$. For induction process, assume the equality holds when $m < m'$. Let $m = m'$ and $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ such that $\epsilon_s e(\mathbf{i}_s) g_m e(\mathbf{i}_t) = \epsilon_k$ for $g_m \in \{\psi_k, \epsilon_k \mid 1 \leq k \leq n-1\}$. By induction, we have $\deg \mathbf{s} - \deg h(\mathbf{s}) = \deg \epsilon_s$. Notice that we have $h(\mathbf{s}) = h(\mathbf{t})$. Hence, by Lemma 4.27 and Lemma 4.28, (4.3) holds, which implies (4.2) holds. Therefore, we have

$$\deg \mathbf{t} - \deg h(\mathbf{t}) = \deg \mathbf{t} - \deg \mathbf{s} + \deg \mathbf{s} - \deg h(\mathbf{s}) = \deg e(\mathbf{i}_s) g_m e(\mathbf{i}_t) + \deg \epsilon_s = \deg \epsilon_k,$$

which completes the induction process. Therefore, the equality (4.5) holds.

Now assume $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. By Lemma 4.17, we have

$$\begin{aligned} \deg \psi_{st} &= \deg \epsilon_s^* \psi_{h(\mathbf{s})h(\mathbf{t})} \epsilon_t = \deg \epsilon_s + \deg \psi_{h(\mathbf{s})h(\mathbf{t})} + \deg \epsilon_t \\ &= \deg \mathbf{s} - \deg h(\mathbf{s}) + \deg h(\mathbf{s}) + \deg h(\mathbf{t}) + \deg \mathbf{t} - \deg h(\mathbf{t}) = \deg \mathbf{s} + \deg \mathbf{t}, \end{aligned}$$

which completes the proof. \square

4.2. The induction property

In Section 4.1 we constructed a set of homogeneous elements $\{\psi_{st}\}$ in $\mathcal{G}_n(\delta)$. Define

$$R_n(\delta) := \{a \in \mathcal{G}_n(\delta) \mid a = \sum_{s, t \in \mathcal{T}_n^{ud}(\lambda)} c_{st} \psi_{st} \text{ where } c_{st} \in R \text{ and } (\lambda, f) \in \widehat{B}_n\}.$$

It is easy to see that $R_n(\delta)$ is a subspaces of $\mathcal{G}_n(\delta)$. In this subsection, we prove the induction property of $R_n(\delta)$. The result of this subsection can be directly implied to $\mathcal{G}_n(\delta)$ after we prove $R_n(\delta) = \mathcal{G}_n(\delta)$ at the end of Section 5.

By the definition of $\mathcal{G}_n(\delta)$, we can consider $\mathcal{G}_n(\delta)$ as a subalgebra of $\mathcal{G}_{n+1}(\delta)$ by identifying $e(\mathbf{i}) = \sum_{k \in P} e(\mathbf{i} \vee k)$ for $\mathbf{i} \in P^n$. Hence we have a sequence

$$\mathcal{G}_1(\delta) \subset \mathcal{G}_2(\delta) \subset \mathcal{G}_3(\delta) \subset \dots$$

For each $i \in P$, define

$$e_{n,i} := \sum_{\mathbf{j} \in P^n} e(\mathbf{j} \vee i) \in \mathcal{G}_{n+1}(\delta).$$

Similar to the cyclotomic Khovnov-Lauda-Rouquier algebras, there is a (non-unital) embedding of $\theta_i^{(n)} : \mathcal{G}_n(\delta) \hookrightarrow \mathcal{G}_{n+1}(\delta)$ given by

$$e(\mathbf{j}) \mapsto e(\mathbf{j} \vee i), \quad y_r \mapsto e_{n,i} y_r, \quad \psi_s \mapsto e_{n,i} \psi_s \quad \text{and} \quad \epsilon_s \mapsto e_{n,i} \epsilon_s,$$

for $\mathbf{j} \in P^n$, $2f+1 \leq r \leq n$ and $2f+1 \leq s \leq n-1$. As $R_n(\delta)$ is a R -subspace of $\mathcal{G}_n(\delta)$, we can restrict $\theta_i^{(n)}$ to $R_n(\delta)$.

First we set up the notations and definitions we are going to use. Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Define

$$R_n^{\geq f}(\delta) := \{a \in \mathcal{G}_n(\delta) \mid a = \sum_{s, t \in \mathcal{T}_n^{ud}(\mu)} c_{st} \psi_{st} \text{ where } c_{st} \in R, (\mu, m) \in \widehat{B}_n \text{ and } m \geq f\},$$

$$R_n^{> f}(\delta) := \{a \in \mathcal{G}_n(\delta) \mid a = \sum_{s, t \in \mathcal{T}_n^{ud}(\mu)} c_{st} \psi_{st} \text{ where } c_{st} \in R, (\mu, m) \in \widehat{B}_n \text{ and } m > f\}.$$

For $a, b \in \mathcal{G}_n(\delta)$, we write $a \equiv b \pmod{R_n^{\geq f}(\delta)}$ if $a - b \in R_n^{\geq f}(\delta)$, and $a \equiv b \pmod{R_n^{> f}(\delta)}$ in a similar way. Next we define a subset of $\bigcup_{i \geq 1} \widehat{B}_i$.

4.30. Definition. Define $\widehat{\mathcal{B}}$ to be the subset of $\bigcup_{i \geq 1} \widehat{B}_i$ such that $(\lambda, f) \in \widehat{\mathcal{B}}$ with $\lambda \vdash n - 2f$ if and only if for any $s, t \in \mathcal{T}_n^{ud}(\lambda)$ and $a \in \mathcal{G}_n(\delta)$, we have

$$\psi_{st} \cdot a \equiv \sum_{v \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{sv} \pmod{R_n^{\geq f}(\delta)}.$$

4.31. Remark. By applying the involution $*$ it is easy to see that the above definition is equivalent to say that $(\lambda, f) \in \widehat{\mathcal{B}}$ with $\lambda \vdash n - 2f$ if and only if for any $s, t \in \mathcal{T}_n^{ud}(\lambda)$ and $b \in \mathcal{G}_n(\delta)$, we have

$$b \cdot \psi_{st} \equiv \sum_{u \in \mathcal{T}_n^{ud}(\lambda)} c_u \psi_{ut} \pmod{R_n^{\geq f}(\delta)}.$$

We define a total ordering on $\bigcup_{i \geq 1} \widehat{B}_i$ extended by the lexicographic ordering on \widehat{B}_n . Suppose $(\lambda, f), (\mu, m) \in \bigcup_{i \geq 1} \widehat{B}_i$. We denote $(\mu, m) \geq (\lambda, f)$ if $|\mu| + 2m < |\lambda| + 2f$; or $|\mu| + 2m = |\lambda| + 2f$ and $m > f$; or $|\mu| + 2m = |\lambda| + 2f$ and $m = f$, and $\mu \geq \lambda$. Define $(\mu, m) > (\lambda, f)$ if $(\mu, m) \geq (\lambda, f)$ and $(\mu, m) \neq (\lambda, f)$.

4.32. Definition. Define $\mathcal{S}_n = \{(\lambda, f) \in \widehat{B}_n \mid (\mu, m) \in \widehat{\mathcal{B}} \text{ whenever } (\mu, m) \in \bigcup_{i \geq 1} \widehat{B}_i \text{ and } (\mu, m) > (\lambda, f)\}.$

The next Lemma is the key point of \mathcal{S}_n .

4.33. Lemma. Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. If there exists $\sigma \vdash n - 2f$ such that $(\sigma, f) \in \mathcal{S}_n$, then $R_n^{\geq f}(\delta)$ is a two-sided $\mathcal{G}_n(\delta)$ -ideal.

Proof. It is directly implied by the definitions of \mathcal{S}_n and $R_n^{\geq f}(\delta)$. □

Now we start to prove the induction property of $R_n(\delta)$. Suppose $(\lambda, f) \in \widehat{B}_{n-1}$. Let $\alpha \in \mathcal{A}(\lambda)$ and $\mu = \lambda \cup \{\alpha\}$. Define $t \in \mathcal{T}_{n-1}^{ud}(\lambda)$ and $s \in \mathcal{T}_n^{ud}(\mu)$ where $s|_{n-1} = t$. Write $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ and $\alpha = (r, \mu_r)$. The next Lemma explores the connection between $\psi_{t\epsilon_1}$ and $\psi_s \epsilon_s$.

4.34. Lemma. Suppose s, t are defined as above. Let $a = 2f + \mu_1 + \mu_2 + \dots + \mu_r$ and $\text{res}(\alpha) = i \in P$. We have $\psi_s = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_i^{(n-1)}(\psi_t)$ and $\epsilon_s = \theta_i^{(n-1)}(\epsilon_t)$.

Proof. By the construction of ϵ_t , one can see that ϵ_t depends on the remove pairs of t only. Because $s(n) = \alpha > 0$, the remove pairs of t and s are the same. Hence we have $\epsilon_s = \theta_i^{(n-1)}(\epsilon_t)$.

We define $u = h(t) \in \mathcal{T}_{n-1}^{ud}(\lambda)$ and $v = h(s) \in \mathcal{T}_n^{ud}(\mu)$. Let $w = t^{(\mu, f)} s_a s_{a+1} \dots s_{n-1}$. By the construction one can see that $w(n) = v(n) = \alpha$ and $w|_{n-1} = t^{(\lambda, f)}$. Because $t = s|_{n-1}$ and $s(n) = \alpha > 0$, we have $u = v|_{n-1}$ and $u(n) = \alpha = w(n)$, which implies

$$v = w \cdot d(u) = t^{(\lambda, f)} s_a s_{a+1} \dots s_{n-1} d(u).$$

Hence we have $\psi_s = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_i^{(n-1)}(\psi_t)$, which completes the proof. \square

4.35. Example. Suppose $n = 8$ and $\delta = 1$. Let $(\lambda, f) = (\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, 2) \in \widehat{B}_{n-1}$, the node $\alpha = (3, 1)$ and $i = \text{res}(\alpha) = 2 \in P$. Define $\mu = \lambda \cup \{\alpha\} = \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$ and the following up-down tableaux

$$\begin{aligned} u &= (\emptyset, \square, \square, \square, \square, \square, \square, \square) \in \mathcal{T}_n^{ud}(\lambda), \\ v &= (\emptyset, \square, \square, \square, \square, \square, \square, \square) \in \mathcal{T}_n^{ud}(\lambda), \\ s &= (\emptyset, \square, \square, \square, \square, \square, \square, \square) \in \mathcal{T}_{n+1}^{ud}(\mu), \\ t &= (\emptyset, \square, \square, \square, \square, \square, \square, \square) \in \mathcal{T}_{n+1}^{ud}(\mu). \end{aligned}$$

One can see that $s|_{n-1} = u$, $t|_{n-1} = v$ and $s(n) = t(n) = \alpha$. By direct calculations, we have $\psi_u = 1$, $\psi_v = \psi_6$, $\psi_s = \psi_7$, $\psi_t = \psi_7 \psi_6$ and

$$\begin{aligned} \epsilon_u &= e(0, 0, 0, 0, 0, 1, -1) \epsilon_4 \epsilon_5 e(0, 0, 0, 1, -1, 1, -1) \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 \psi_6 e(0, 1, -1, 1, 2, -1, -2), \\ \epsilon_v &= e(0, 0, 0, 0, 0, -1, 1) \epsilon_4 \epsilon_5 \epsilon_6 e(0, 0, 0, -1, 1, 2, -2) \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 e(0, -1, 1, 2, 3, -3, -2), \\ \epsilon_s &= e(0, 0, 0, 0, 0, 1, -1, 2) \epsilon_4 \epsilon_5 e(0, 0, 0, 1, -1, 1, -1, 2) \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 \psi_6 e(0, 1, -1, 1, 2, -1, -2, 2), \\ \epsilon_t &= e(0, 0, 0, 0, 0, -1, 1, 2) \epsilon_4 \epsilon_5 \epsilon_6 e(0, 0, 0, -1, 1, 2, -2, 2) \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 e(0, -1, 1, 2, 3, -3, -2, 2). \end{aligned}$$

Because $\mu = (3, 1)$, let $a = 2f + \mu_1 = 7$. Hence we have $\psi_s = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_i^{(n-1)}(\psi_u) = \psi_7 \theta_i^{(n-1)}(\psi_u)$, $\psi_t = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_i^{(n-1)}(\psi_v) = \psi_7 \theta_i^{(n-1)}(\psi_v)$, $\epsilon_s^* = \theta_i^{(n-1)}(\epsilon_u^*)$ and $\epsilon_t = \theta_i^{(n-1)}(\epsilon_v)$.

The following two results give the induction property of $R_n(\delta)$.

4.36. Lemma. Suppose $(\lambda, f) \in \widehat{B}_{n-1}$. If $i \in P$ such that $i = \text{res}(\alpha)$ for some $\alpha \in \mathcal{A}(\lambda)$, then we have $\theta_i^{(n-1)}(\psi_{uv}) = \psi_{st}$ for any $u, v \in \mathcal{T}_{n-1}^{ud}(\lambda)$. Moreover, we have $s|_{n-1} = u$, $t|_{n-1} = v$ and $s(n) = t(n) = \alpha$.

Proof. First we show that when $u = v = t^{(\lambda, f)}$, we have $\theta_i^{(n-1)}(\psi_{uv}) = \psi_{xx}$ where $x|_{n-1} = t^{(\lambda, f)}$ and $x(n) = \alpha$.

Set $\mu = \lambda \cup \{\alpha\} = (\mu_1, \mu_2, \dots, \mu_m)$. One can see that $\alpha = (r, \mu_r)$ for some $1 \leq r \leq m$. Hence let $a = 2f + \mu_1 + \mu_2 + \dots + \mu_r$. We have a reduced expression $d(x) = s_a s_{a+1} \dots s_{n-1} \in \mathfrak{S}_{2f, n+1}$.

Let $\mathbf{i}_{(\lambda, f)} = (i_1, i_2, \dots, i_{n-1})$. By the construction of λ and μ , as $i = \text{res}(\alpha)$, we have $|i - i_s| \geq 2$ for any $a \leq s \leq n-1$ and $\mathbf{i}_{(\mu, f)} = (i_1, \dots, i_{a-1}, i, i_a, i_{a+1}, \dots, i_{n-1})$. Hence by (3.14) we have

$$\begin{aligned} \theta_i^{(n-1)}(\psi_{t^{(\lambda, f)} t^{(\lambda, f)}}) &= e(\mathbf{i}_{(\lambda, f)} \vee i) \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} e(\mathbf{i}_{(\lambda, f)} \vee i) \\ &= e(\mathbf{i}_{(\lambda, f)} \vee i) \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \psi_n \psi_{n-1} \dots \psi_a e(\mathbf{i}_{(\mu, f)}) \psi_a \dots \psi_{n-1} \psi_n \\ &= \psi_n \psi_{n-1} \dots \psi_a e(\mathbf{i}_{(\mu, f)}) \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} e(\mathbf{i}_{(\mu, f)}) \psi_a \dots \psi_{n-1} \psi_n \\ &= \psi_{d(x)}^* e_{(\mu, f)} \psi_{d(x)} = \psi_{xx}. \end{aligned}$$

Now for any $u, v \in \mathcal{T}_{n-1}^{ud}(\lambda)$, set s, t to be up-down tableaux such that $s|_{n-1} = u$, $t|_{n-1} = v$ and $s(n) = t(n) = \alpha$. By Lemma 4.34, we have $\psi_s = \psi_{d(x)} \theta_i^{(n-1)}(\psi_u)$, $\psi_t = \psi_{d(x)} \theta_i^{(n-1)}(\psi_v)$, $\epsilon_s^* = \theta_i^{(n-1)}(\epsilon_u^*)$ and $\epsilon_t = \theta_i^{(n-1)}(\epsilon_v)$. Hence

$$\begin{aligned} \theta_i^{(n-1)}(\psi_{uv}) &= \theta_i^{(n-1)}(\epsilon_u^* \psi_u^* e_{(\lambda, f)} \psi_v \epsilon_v) = \theta_i^{(n-1)}(\epsilon_u^* \psi_u^* \theta_i^{(n-1)}(e_{(\lambda, f)}) \theta_i^{(n-1)}(\psi_v \epsilon_v) = \theta_i^{(n-1)}(\epsilon_u^* \psi_u^*) \psi_{xx} \theta_i^{(n-1)}(\psi_v \epsilon_v) \\ &= \theta_i^{(n-1)}(\epsilon_u^*) \theta_i^{(n-1)}(\psi_u^*) \psi_x^* e_{(\mu, f)} \psi_x \theta_i^{(n-1)}(\psi_v) \theta_i^{(n-1)}(\epsilon_v) = \epsilon_s^* \psi_s^* e_{(\mu, f)} \psi_t \epsilon_t = \psi_{st}, \end{aligned}$$

which completes the proof. \square

4.37. Lemma. Suppose $(\lambda, f) \in \widehat{B}_{n-1}$. If there exists $\sigma \vdash n - 2f$ such that $(\sigma, f) \in \mathcal{S}_n$, for any $u, v \in \mathcal{T}_{n-1}^{ud}(\lambda)$ and $i \in P$, we have the following results:

- (1) If $\text{res}(\alpha) = i$ for some $\alpha \in \mathcal{A}(\lambda)$, we have $\theta_i^{(n-1)}(\psi_{uv}) y_n \in R_n^{>f}(\delta)$.
- (2) If $\text{res}(\alpha) \neq i$ for all $\alpha \in \mathcal{A}(\lambda)$, we have $\theta_i^{(n-1)}(\psi_{uv}) \in R_n^{>f}(\delta)$.

Proof. Because $\theta_i^{(n-1)}(\psi_{uv}) = \theta_i^{(n-1)}(\epsilon_u^* \psi_u^*) \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)}) \theta_i^{(n-1)}(\psi_{v\epsilon_v})$ and $\theta_i^{(n-1)}(\psi_{v\epsilon_v})$ commutes with y_n , by Lemma 4.33 it suffices to prove the Lemma when $u = v = t^{(\lambda,f)}$.

We prove the Lemma by applying induction twice. First we apply induction on n . The base step is $n = 1$, which is trivial by (3.8). Assume that there exists n' such that the Lemma holds when $n < n'$ and we set $n = n'$. Then we apply the induction on f . The base step is $f = \lfloor \frac{n}{2} \rfloor$, which can be directly verified by (3.14), (3.17), (3.19) in Case (1) and (3.14) in Case (2). We omit the detailed proof here.

Assume that there exists f' such that the Lemma holds when $f < f'$ and we set $f = f'$. Write $\mathbf{i}_{(\lambda,f)} = (i_1, i_2, \dots, i_{n-1})$, $t^{(\lambda,f)} = (\alpha_1, \dots, \alpha_{n-1})$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$.

Here we give two results implied by the induction. These two results will be used in the rest of the proof.

As $(\sigma, f) \in \mathcal{S}_n$, by Lemma 4.33, $R_n^{>f}(\delta)$ is a two-sided $\mathcal{G}_n(\delta)$ -ideal. Hence by induction on f , we have $\theta_i^{(n-1)}(R_{n-1}^{>f}(\delta)) \subset R_n^{>f}(\delta)$ for any $i \in P$. Moreover, by induction on n , we have $\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1} \in R_{n-1}^{>f}(\delta)$, which yields

$$\theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1}) \in R_n^{>f}(\delta). \quad (4.6)$$

In $\mathcal{G}_{2f+2}(\delta)$, by (3.8) we have $\epsilon_1 \epsilon_3 \dots \epsilon_{2f+1} = \psi_{t^{(0,f+1)}t^{(0,f+1)}} \in R_{2f+2}^{>f}(\delta)$. Therefore, by induction on n and f , we have

$$\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \epsilon_{2f+1} = \sum_{(i_{2f+3}, \dots, i_n) \in P^{n-2f-2}} \theta_{i_n}^{(n-1)} \circ \theta_{i_{n-1}}^{(n-2)} \circ \dots \circ \theta_{i_{2f+3}}^{(2f+2)} (\psi_{t^{(0,f+1)}t^{(0,f+1)}}) \in R_n^{>f}(\delta),$$

which can be generalized by (3.24) and Lemma 4.33:

$$\epsilon_1 \dots \epsilon_{2f-1} \epsilon_\ell = \epsilon_\ell \epsilon_{\ell-1} \dots \epsilon_{2f+2} \cdot (\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \epsilon_{2f+1}) \cdot \epsilon_{2f+2} \dots \epsilon_{\ell-1} \epsilon_\ell \in R_n^{>f}(\delta) \quad (4.7)$$

for any $2f+1 \leq \ell \leq n$.

Now we complete the induction process of f by considering different values of i . For convenience we write $\mathbf{i} = (i_1, \dots, i_n) = \mathbf{i}_{(\lambda,f)} \vee i \in P^n$.

(1). Suppose $\text{res}(\alpha) = i$ for some $\alpha \in \mathcal{A}(\lambda)$. We consider the following 3 cases. Note that $i = i_{n-1}$ is excluded because it is impossible to find $\alpha \in \mathcal{A}(\lambda)$ such that $\text{res}(\alpha) = i = i_{n-1}$.

Case 1.1: $i = -i_{n-1}$.

When $i = -\frac{1}{2}$, because $\text{res}(\alpha) = -\frac{1}{2}$ for some $\alpha \in \mathcal{A}(\lambda)$ and $\text{res}(\alpha_{n-1}) = \frac{1}{2}$ where $\alpha_{n-1} \in \mathcal{B}(\lambda)$, we have $h_n(\mathbf{i}) = -2$, which implies $h_{n-1}(\mathbf{i}) = -1$ by (3.2). Hence we have $\mathbf{i} \in I_{n-1,0}^n$. Therefore, by (3.17) we have

$$\begin{aligned} \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n) &= \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n e(\mathbf{i})) = \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1} e(\mathbf{i})) - 2\theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1} e(\mathbf{i})) \epsilon_{n-1} e(\mathbf{i}) \\ &= \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1}) - 2\theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1}) \epsilon_{n-1} e(\mathbf{i}), \end{aligned}$$

which yields $\theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n) \in R_n^{>f}(\delta)$ by (4.6) and Lemma 4.33.

When $i = \frac{1}{2}$, following the same argument as $i = -\frac{1}{2}$, we have $h_{n-1}(\mathbf{i}) = -1$ and $\mathbf{i} \in I_{n-1,+}^n$. Hence by (3.15) we have

$$\begin{aligned} \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n) &= \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n e(\mathbf{i})) \\ &= (-1)^{a_{n-1}(\mathbf{i})+1} e(\mathbf{i}) \epsilon_1 \dots \epsilon_{2f-1} \epsilon_{n-1} e(\mathbf{i}) + \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1}), \end{aligned}$$

which yields $\theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n) \in R_n^{>f}(\delta)$ by (4.6) and (4.7).

When $i \neq \pm \frac{1}{2}$, following the same argument as when $i = -\frac{1}{2}$, we have $h_{n-1}(\mathbf{i}) = 0$ and $\mathbf{i} \in I_{n-1,+}^n$. By the similar argument as when $i = \frac{1}{2}$, we have $\theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n) \in R_n^{>f}(\delta)$.

Case 1.2: $i = i_{n-1} \pm 1$.

First we consider the case when $i = i_{n-1} - 1$. Notice that when $i = -\frac{1}{2}$, we have $i = -i_{n-1} = -\frac{1}{2}$, which has already been proved in Case 1.1. Hence we set $i \neq -\frac{1}{2}$.

Because $\text{res}(\alpha) = i$ for some $\alpha \in \mathcal{A}(\lambda)$ and $\text{res}(\alpha_{n-1}) = i+1$ where $\alpha_{n-1} \in [\lambda]$, we have $\lambda_m = 1$ by the construction of λ . Let $\mu = \lambda|_{n-2}$ and $\mathbf{j} = \mathbf{i}_{(\mu,f)} = \mathbf{i}_{(\lambda,f)}|_{n-2}$. By (3.14) we have

$$\theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_n) = \theta_i^{(n-1)}(\psi_{t(\lambda,f)t(\lambda,f)} y_{n-1}) - \psi_{n-1} e(\mathbf{j}, i, i+1) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j}, i, i+1) \psi_{n-1}.$$

Because $\lambda_m = 1$, we have $\text{res}(\alpha) \neq i$ for any $\alpha \in \mathcal{A}(\mu)$. Hence by induction on n , we have

$$e(\mathbf{j}, i, i+1) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j}, i, i+1) = \theta_{i+1}^{(n-1)}(\theta_i^{(n-2)}(\psi_{t(\mu,f)t(\mu,f)})) \in \theta_{i+1}^{(n-1)}(R_{n-1}^{>f}(\delta)) \subset R_n^{>f}(\delta),$$

and the Lemma holds by (4.6). Following the similar argument, the Lemma holds when $i = i_{n-1} + 1$.

Case 1.3: $|i - i_{n-1}| > 1$.

This case can be verified using (3.13) and (3.14) directly and we omit the detailed proof here.

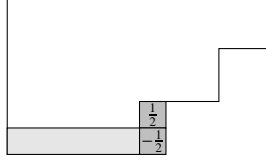
(2). Suppose $\text{res}(\alpha) \neq i$ for all $\alpha \in \mathcal{A}(\lambda)$. We consider the following 5 cases:

Case 2.1: $i = -i_{n-1}$.

When $i = -\frac{1}{2}$, as $\text{res}(\alpha) \neq i$ for any $\alpha \in \mathcal{A}(\lambda)$, we have $h_n(\mathbf{i}) = -1$, which implies $h_{n-1}(\mathbf{i}) = -2$ by (3.2) and hence $\mathbf{i} \in I_{n-1,-}^n$. Therefore, by (3.19) we have

$$\theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) = (-1)^{a_{n-1}(\mathbf{i})} \psi_{\mathbf{tt}} y_{n-1} + (-1)^{a_{n-1}(\mathbf{i})} y_{n-1} \psi_{\mathbf{tt}} \in R_n^{>f}(\delta).$$

When $i = \frac{1}{2}$, as $\text{res}(\alpha) \neq i$ for any $\alpha \in \mathcal{A}(\lambda)$, we have $\lambda_{m-1} = \lambda_m$. See the next diagram for λ :



In the above diagram, the entries in the shadowed nodes are their residues and the residues of the light shadowed nodes are less than $-\frac{1}{2}$. Formally, we set $a = n - \lambda$. So the node α_a is the shadowed node with residue $\frac{1}{2}$ and the node α_{n-1} is the shadowed node with residue $-\frac{1}{2}$. Moreover, for any $a < s < n - 1$, we have $i_a + i_s \neq 0$ and $|i_a - i_s| > 1$. Therefore, by (3.14) and (3.9) we have

$$\theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) = \psi_a \dots \psi_{n-4} \psi_{n-3} e(\mathbf{j}) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j}) \psi_{n-3} \psi_{n-4} \dots \psi_a,$$

where $\mathbf{j} = (i_1, \dots, i_{a-1}, i_{a+1}, \dots, i_{n-2}, i_a, i_{n-1}, i)$. We note that $i_a = i = \frac{1}{2}$ and $i_{n-1} = -\frac{1}{2}$. Hence apply (3.18) to the above equation and we have

$$\begin{aligned} \theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) &= (-1)^{a_{n-1}(\mathbf{i})} \psi_a \dots \psi_{n-4} \psi_{n-3} e(\mathbf{j}) \epsilon_1 \dots \epsilon_{2f-1} \epsilon_{n-1} e(\mathbf{j}) \psi_{n-3} \psi_{n-4} \dots \psi_a \\ &\quad - 2(-1)^{a_{n-2}(\mathbf{i})} \psi_a \dots \psi_{n-4} \psi_{n-3} e(\mathbf{j}) \epsilon_1 \dots \epsilon_{2f-1} \epsilon_{n-2} e(\mathbf{j}) \psi_{n-3} \psi_{n-4} \dots \psi_a \\ &\quad + \psi_a \dots \psi_{n-4} \psi_{n-3} e(\mathbf{j}) \epsilon_1 \dots \epsilon_{2f-1} \epsilon_{n-1} \epsilon_{n-2} e(\mathbf{j}) \psi_{n-3} \psi_{n-4} \dots \psi_a \\ &\quad + \psi_a \dots \psi_{n-4} \psi_{n-3} e(\mathbf{j}) \epsilon_1 \dots \epsilon_{2f-1} \epsilon_{n-2} \epsilon_{n-1} e(\mathbf{j}) \psi_{n-3} \psi_{n-4} \dots \psi_a, \end{aligned}$$

which yields $\theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) \in R_n^{>f}(\delta)$ by (4.7) and Lemma 4.33.

When $i \neq \pm\frac{1}{2}$, as $\text{res}(\alpha) \neq i$ for any $\alpha \in \mathcal{A}(\lambda)$, we have $h_{n-1}(\mathbf{i}) = -1$ and $\mathbf{i} \in I_{n-1,0}^n$. By (3.15) and (4.7), we have

$$\theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) = (-1)^{a_{n-1}(\mathbf{i})} e(\mathbf{i}) \epsilon_1 \dots \epsilon_{2f-1} \epsilon_{n-1} e(\mathbf{i}) \in R_n^{>f}(\delta).$$

Case 2.2: $i = i_{n-1}$.

When $i = i_{n-1} = 0$, we have $\mathbf{i} \in I_{n-1,0}^n$. Hence following the same argument as in Case 2.1 when $i \neq \pm\frac{1}{2}$, we have $\theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) \in R_n^{>f}(\delta)$.

When $i = i_{n-1} \neq 0$, it is easy to verify the following equality holds by (3.12), (3.13) and (3.14):

$$e(\mathbf{i}) = -\psi_{n-1} e(\mathbf{i}) y_{n-1}^2 \psi_{n-1} - e(\mathbf{i}) y_{n-1} \psi_{n-1} - \psi_{n-1} e(\mathbf{i}) y_{n-1}.$$

Hence by (4.6), we have

$$\theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) = -\psi_{n-1} \theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)} y_{n-1}^2) \psi_{n-1} - \theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)} y_{n-1}) \psi_{n-1} - \psi_{n-1} \theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)} y_{n-1}) \in R_n^{>f}(\delta).$$

Case 2.3: $i = i_{n-1} - 1$.

The case $i = -\frac{1}{2}$ has been proved in Case 2.1. Hence we assume $i \neq -\frac{1}{2}$. As $\text{res}(\alpha) \neq i$ for any $\alpha \in \mathcal{A}(\lambda)$, we have $\lambda_m > 1$. Hence we have $i_{n-1} = i + 1$ and $i_{n-2} = i_{n-1} - 1 = i$. Define $\mathbf{j} = (i_1, i_2, \dots, i_{n-3})$. By (3.31) we have

$$\begin{aligned} \theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) &= \psi_{n-2} \psi_{n-1} e(\mathbf{j} \vee i, i, i+1) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j} \vee i, i, i+1) \psi_{n-2} \\ &\quad - \psi_{n-1} \psi_{n-2} e(\mathbf{j} \vee i+1, i, i) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j} \vee i+1, i, i) \psi_{n-1}. \end{aligned} \quad (4.8)$$

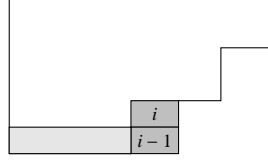
Let $\mu = \lambda|_{n-2}$ and $\gamma = \lambda|_{n-3}$. One can see that $\text{res}(\alpha) \neq i$ for any $\alpha \in \mathcal{A}(\mu)$ and $\text{res}(\alpha) \neq i+1$ for any $\alpha \in \mathcal{A}(\gamma)$ by the construction of λ . Hence by induction on n , we have

$$\begin{aligned} e(\mathbf{j} \vee i, i, i+1) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j} \vee i, i, i+1) &= \theta_{i+1}^{(n-1)}(\theta_i^{(n-2)}(\psi_{\mathbf{t}(\mu,f)\mathbf{t}(\mu,f)})) \in \theta_{i+1}^{(n-1)}(R_{n-1}^{>f}(\delta)) \subset R_n^{>f}(\delta), \\ e(\mathbf{j} \vee i+1, i, i) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j} \vee i+1, i, i) &= \theta_i^{(n-1)}(\theta_{i+1}^{(n-2)}(\theta_{i+1}^{(n-3)}(\psi_{\mathbf{t}(\gamma,f)\mathbf{t}(\gamma,f)}))) \in \theta_i^{(n-1)}(\theta_{i+1}^{(n-2)}(R_{n-2}^{>f}(\delta))) \subset R_n^{>f}(\delta). \end{aligned}$$

Substitute the above equalities to (4.8). By Lemma 4.33, we have $\theta_i^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) \in R_n^{>f}(\delta)$.

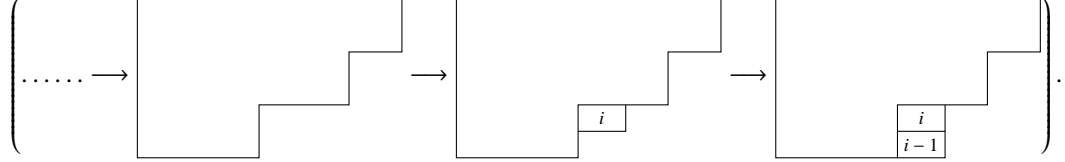
Case 2.4: $i = i_{n-1} + 1$.

The case $i = \frac{1}{2}$ has been proved in Case 2.1. Hence we assume $i \neq \frac{1}{2}$. As $\text{res}(\alpha) \neq i$ for any $\alpha \in \mathcal{A}(\lambda)$, we have $\lambda_{m-1} = \lambda_m$. See the next diagram for λ :



In the above diagram, the entries in the shadowed nodes are their residues and the residues of the light shadowed nodes are less than $i - 1$. Formally, we set $a = n - \lambda$. So the node α_a is the shadowed node with residue i and the node α_{n-1} is the shadowed node with residue $i - 1$. Moreover, for any $a < s < n - 1$, we have $|i_a - i_s| > 1$.

Notice that as $\lambda_{m-1} = \lambda_m$, we have $t^{(\lambda, f)} s_a s_{a+1} \dots s_{n-3} \in \mathcal{T}_{n-1}^{ud}(\lambda)$, which is of the form



By Corollary 4.14 and (3.9), we have

$$\theta_i^{(n-1)}(\psi_{t^{(\lambda, f)} t^{(\lambda, f)}}) = \psi_a \psi_{a+1} \dots \psi_{n-3} e(\mathbf{j}) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j}) \psi_{n-3} \dots \psi_{a+1} \psi_a,$$

where $\mathbf{j} = (i_1, \dots, i_{a-1}, i_{a+1}, \dots, i_{n-2}, i_a, i_{n-1}, i)$. We note that $(i_a, i_{n-1}, i) = (i, i - 1, i)$. Hence, following the similar argument as in Case 2.3, we have $e(\mathbf{j}) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{j}) \in R_n^{>f}(\delta)$. Therefore we have $\theta_i^{(n-1)}(\psi_{t^{(\lambda, f)} t^{(\lambda, f)}}) \in R_n^{>f}(\delta)$ by Lemma 4.33.

Case 2.5: $|i - i_n| > 1$.

This case can be verified by (3.14) directly and we omit the detailed proof here. \square

The next Lemma combines the results of Lemma 4.36 and Lemma 4.37.

4.38. Lemma. Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. If there exists $\sigma \vdash n - 2f$ such that $(\sigma, f) \in \mathcal{S}_n$, for any $i \in P$ we have $\theta_i^{(n-1)}(R_{n-1}^{\geq f}(\delta)) \subset R_n^{\geq f}(\delta)$ and $\theta_i^{(n-1)}(R_{n-1}^{>f}(\delta)) \subset R_n^{>f}(\delta)$. Moreover, if $\mathcal{S}_n = \widehat{B}_n$, then for any $i \in P$ we have $\theta_i^{(n-1)}(R_{n-1}(\delta)) \subseteq R_n(\delta)$.

Proof. Suppose $(\mu, m) \in \widehat{B}_{n-1}$ and $s, t \in \mathcal{T}_{n-1}^{ud}(\mu)$. When $m = f$, if $\text{res}(\alpha) = i$ for some $\alpha \in \mathcal{A}(\mu)$, by Lemma 4.36 we have $\theta_i^{(n-1)}(\psi_{st}) \in R_n^{>f}(\delta)$; and if $\text{res}(\alpha) \neq i$ for all $\alpha \in \mathcal{A}(\mu)$, by Lemma 4.37 we have $\theta_i^{(n-1)}(\psi_{st}) \in R_n^{>f}(\delta)$. Hence we have

$$\theta_i^{(n-1)}(\psi_{st}) \in R_n^{>f}(\delta). \quad (4.9)$$

When $m > f$, by Lemma 4.36 and Lemma 4.37, we have

$$\theta_i^{(n-1)}(\psi_{st}) \in R_n^{>f}(\delta). \quad (4.10)$$

By (4.9) and (4.10), we have $\theta_i^{(n-1)}(R_{n-1}^{\geq f}(\delta)) \subset R_n^{\geq f}(\delta)$ and $\theta_i^{(n-1)}(R_{n-1}^{>f}(\delta)) \subset R_n^{>f}(\delta)$.

Suppose $\mathcal{S}_n = \widehat{B}_n$. Choose λ such that $(\lambda, 0) \in \widehat{B}_n$. Hence that $(\lambda, 0) \in \mathcal{S}_n$ and we have $\theta_i^{(n-1)}(R_{n-1}^{\geq 0}(\delta)) \subseteq R_n^{\geq 0}(\delta)$. Notice that $R_n(\delta) = R_n^{\geq 0}(\delta)$. Therefore we have $\theta_i^{(n-1)}(R_{n-1}(\delta)) \subseteq R_n(\delta)$. \square

Recall by identifying $e(\mathbf{i}) = \sum_{i \in P} e(\mathbf{i} \vee i)$ for $\mathbf{i} \in P^{n-1}$, we consider $\mathcal{G}_{n-1}(\delta)$ as a subalgebra of $\mathcal{G}_n(\delta)$ and obtain a sequence

$$\mathcal{G}_1(\delta) \subset \mathcal{G}_2(\delta) \subset \mathcal{G}_3(\delta) \subset \dots$$

The key point of Lemma 4.38 is that by assuming $\mathcal{S}_n = \widehat{B}_n$, we can construct such sequence for $R_n(\delta)$ as well. In Lemma 4.38, as i is chosen arbitrary in P , we can consider $R_{n-1}(\delta)$ as a subspace of $R_n(\delta)$ by identifying $e(\mathbf{i}) = \sum_{i \in P} e(\mathbf{i} \vee i)$ for $\mathbf{i} \in P^{n-1}$. Hence we obtain a sequence

$$R_1(\delta) \subset R_2(\delta) \subset R_3(\delta) \subset \dots \subset R_n(\delta).$$

The next Proposition is the most important application of Lemma 4.38 in this paper.

4.39. Proposition. Suppose $\bigcup_{i=1}^n \widehat{B}_i \subseteq \widehat{\mathcal{B}}$. Then $e(\mathbf{i}) \in R_n(\delta)$ for any $\mathbf{i} \in P^n$.

Proof. Because $\mathcal{S}_n = \widehat{B}_n$, by the definition of \mathcal{S}_n we have $\mathcal{S}_k = \widehat{B}_k$ for any $1 \leq k \leq n$. Therefore, if we write $\mathbf{i} = (i_1, i_2, \dots, i_n)$, as $i_1 = \frac{\delta-1}{2}$ and $e(i_1) = 1 \in R_1(\delta) \subset \mathcal{G}_1(\delta)$, we have

$$e(\mathbf{i}) = \theta_{i_n}^{(n-1)} \circ \theta_{i_{n-1}}^{(n-2)} \circ \dots \circ \theta_{i_2}^{(1)}(1) \in R_n(\delta)$$

by Lemma 4.38. \square

Suppose $\bigcup_{i=1}^n \widehat{B}_i \subseteq \widehat{\mathcal{B}}$. Proposition 4.39 shows that $1 \in R_n(\delta)$. By the definition of $\widehat{\mathcal{B}}$, $R_n(\delta)$ is a right $\mathcal{G}_n(\delta)$ -module. Hence, $\bigcup_{i=1}^n \widehat{B}_i \subseteq \widehat{\mathcal{B}}$ implies $\{\psi_{\mathbf{s}\mathbf{t}} \mid \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda), (\lambda, f) \in \widehat{B}_n\}$ is a R -spanning set of $\mathcal{G}_n(\delta)$, and it has cellular-like property by the definition of $\widehat{\mathcal{B}}$.

Finally we introduce some applications of induction property of $R_n(\delta)$.

4.40. Lemma. Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. If there exists $\sigma \vdash n - 2f$ such that $(\sigma, f) \in \mathcal{S}_n$, we have $\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \epsilon_k \in R_n^{>f}(\delta)$ for $2f + 1 \leq k \leq n - 1$.

Proof. It suffices to prove that when $k = 2f + 1$ the Lemma holds, because when $k > 2f + 1$, by (3.24) we have

$$\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \epsilon_k = \epsilon_k \epsilon_{k-1} \dots \epsilon_{2f+2} \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \epsilon_{2f+1} \epsilon_{2f+2} \dots \epsilon_{k-1} \epsilon_k,$$

and $R_n^{>f}(\delta)$ is a two-sided $\mathcal{G}_n(\delta)$ -ideal by Lemma 4.33.

Consider $k = 2f + 1$. By (3.8) we have

$$\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \epsilon_{2f+1} = \sum_{(i_{2f+3}, \dots, i_n) \in P^{n-2f-2}} \theta_{i_n}^{(n-1)} \circ \theta_{i_{n-1}}^{(n-2)} \circ \dots \circ \theta_{i_{2f+3}}^{(2f+2)} (\psi_{\mathbf{t}(\mathbf{0}, f+1) \mathbf{t}(\mathbf{0}, f+1)}).$$

Because $\psi_{\mathbf{t}(\mathbf{0}, f+1) \mathbf{t}(\mathbf{0}, f+1)} \in R_{2f+2}^{>f}(\delta)$, the Lemma follows by Lemma 4.38. \square

4.41. Lemma. Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. If there exists $\sigma \vdash n - 2f$ such that $(\sigma, f) \in \mathcal{S}_n$, then we have $e(\mathbf{i}) = 0$ if $h_k(\mathbf{i}) > 0$ for some $1 \leq k \leq n - 1$.

Proof. Let $\mathbf{j} = \mathbf{i}|_{n-1} \in P^{n-1}$. As $(\sigma, f) \in \mathcal{S}_n$, we have $\bigcup_{i=1}^{n-1} \widehat{B}_i \subseteq \widehat{\mathcal{B}}$, which implies $e(\mathbf{j}) \in R_{n-1}(\delta)$ by Proposition 4.39. If $h_k(\mathbf{i}) > 0$ for some $1 \leq k \leq n - 1$, by Lemma 3.6 we have $\mathbf{j} \notin I^{n-1}$. Hence by Lemma 4.10 we have $e(\mathbf{j}) = 0$. Therefore $e(\mathbf{i}) = \theta_{i_n}^{(n-1)}(e(\mathbf{j})) = 0$. \square

4.42. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. Let $1 \leq k \leq n$. If $\mathbf{t}(r) > 0$ for all $k \leq r \leq n$, then for any $a \in \mathcal{G}_{k-1}(\delta)$, we have

$$\psi_{\mathbf{t}(\lambda, f) \mathbf{t}} a \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathbf{v}} \psi_{\mathbf{t}(\lambda, f) \mathbf{v}} \pmod{R_n^{>f}(\delta)},$$

where $c_{\mathbf{v}} \neq 0$ only if $\mathbf{v}(r) = \mathbf{t}(r)$ for any $k \leq r \leq n$.

Proof. We only prove the case when $k = n - 1$. For smaller k the proof is essentially the same.

Suppose the head of \mathbf{t} is h and $\lambda = (\lambda_1, \dots, \lambda_m)$. Define $\mathbf{t} = \mathbf{t}|_{n-1}$ and $\mu = \mathbf{t}_{n-1}$. As $\mathbf{t}(n) > 0$, we have $\mathbf{t}(n) = \alpha$ for some $\alpha \in \mathcal{A}(\mu)$ and $\lambda = \mu \cup \{\alpha\}$. Hence $\alpha = (\ell, \lambda_\ell)$ for some $1 \leq \ell \leq m$. Let $a = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$ and $\text{res}(\alpha) = i_n$. By Lemma 4.34, we have $\psi_{\mathbf{t}} = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_{i_n}^{(n-1)}(\psi_{\mathbf{t}})$ and $\epsilon_1 = \theta_{i_n}^{(n-1)}(\epsilon_1)$. Hence by Lemma 4.36, we have

$$\psi_{\mathbf{t}(\lambda, f) \mathbf{t}} a = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_{i_n}^{(n-1)}(\psi_{\mathbf{t}(\mu, f) \mathbf{t}} a).$$

As $(\lambda, f) \in \mathcal{S}_n$, by Lemma 4.38 we have

$$\psi_{\mathbf{t}(\lambda, f) \mathbf{t}} a = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_{i_n}^{(n-1)}(\psi_{\mathbf{t}(\mu, f) \mathbf{t}} a) \equiv \sum_{\mathbf{v} \in \mathcal{T}_{n-1}^{ud}(\mu)} c_{\mathbf{v}} \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_{i_n}^{(n-1)}(\psi_{\mathbf{t}(\mu, f) \mathbf{v}}) \pmod{R_n^{>f}(\delta)}. \quad (4.11)$$

For $\mathbf{v} \in \mathcal{T}_{n-1}^{ud}(\mu)$, define $\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{v}|_{n-1} = \mathbf{v}$ and $\mathbf{v}(n) = \alpha$. By Lemma 4.34 we have $\psi_{\mathbf{v}} = \psi_a \psi_{a+1} \dots \psi_{n-1} \theta_{i_n}^{(n-1)}(\psi_{\mathbf{v}})$ and $\epsilon_{\mathbf{v}} = \theta_{i_n}^{(n-1)}(\epsilon_{\mathbf{v}})$. Hence by Lemma 4.36, we have

$$\psi_a \psi_{a+1} \dots \psi_{n-1} \theta_{i_n}^{(n-1)}(\psi_{\mathbf{t}(\mu, f) \mathbf{v}}) = e_{(\lambda, f)} \psi_{\mathbf{v}} \epsilon_{\mathbf{v}} = \psi_{\mathbf{t}(\lambda, f) \mathbf{v}}, \quad (4.12)$$

where $\mathbf{v}(n) = \alpha = \mathbf{t}(n)$. The Lemma holds by substituting (4.12) into (4.11). \square

4.3. The restriction property

In this subsection we introduce the restriction property of $\mathcal{G}_n(\delta)$. Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Define $P_{f,n} = \{\mathbf{i} \in P^n \mid i_1 = i_3 = \dots = i_{2f+1} = -i_2 = -i_4 = \dots = -i_{2f} = \frac{\delta-1}{2}\}$ and $\mathcal{G}_{2f,n}(\delta)$ as the subalgebra of $\mathcal{G}_n(\delta)$ generated by

$$G_{f,n}(\delta) = \{e(\mathbf{i}) \mid \mathbf{i} \in P_{f,n}\} \cup \{y_k \mid 2f+1 \leq k \leq n\} \cup \{\psi_k \mid 2f+1 \leq k \leq n-1\} \cup \{\epsilon_k \mid 2f+1 \leq k \leq n-1\}.$$

Denote $\epsilon_{1,0} = 1$ and $\epsilon_{1,f} = \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1}$ for $f > 0$. We define a map $\phi_{f,n}^{(g)} : G_{f,n}(\delta) \rightarrow G_{n-2f}(\delta)$ by

$$e(\mathbf{i}) \mapsto e(i_{2f+1}, i_{2f+2}, \dots, i_n), \quad y_r \mapsto y_{r-2f}, \quad \psi_s \mapsto \psi_{s-2f}, \quad \text{and} \quad \epsilon_s \mapsto \epsilon_{s-2f},$$

where $\mathbf{i} \in P_{f,n}$, $2f+1 \leq r \leq n$ and $2f+1 \leq s \leq n-1$. Extend $\phi_{f,n}^{(g)}$ to a linear map $\phi_{f,n} : \epsilon_{1,f} \mathcal{G}_{2f,n}(\delta) \rightarrow \mathcal{G}_{n-2f}(\delta)$ such that for each $a \in \mathcal{G}_{2f,n}(\delta)$, if we can write $a = g_1 g_2 \dots g_k$ where $g_i \in G_{f,n}(\delta)$ for $1 \leq i \leq k$, then

$$\phi_{f,n}(\epsilon_{1,f} a) = \phi_{f,n}^{(g)}(g_1) \phi_{f,n}^{(g)}(g_2) \dots \phi_{f,n}^{(g)}(g_k) \in \mathcal{G}_{n-2f}(\delta).$$

4.43. Lemma. Suppose $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. The linear map $\phi_{f,n} : \epsilon_{1,f} \mathcal{G}_{2f,n}(\delta) \rightarrow \mathcal{G}_{n-2f}(\delta)$ is well-defined.

Proof. When $f = 0$, $\phi_{0,n}$ is the identity map. Hence we assume $f > 0$. It suffices to check the relations of $\epsilon_{1,f} \mathcal{G}_{2f,n}(\delta)$ consist in $\mathcal{G}_{n-2f}(\delta)$ by applying $\phi_{f,n}$. Let $\mathbf{i} = (i_1, \dots, i_n) \in P_{f,n}$ and $\mathbf{j} = (j_1, \dots, j_n) \in P_{f,n}$. By direct calculation, we have $h_k(\mathbf{i}) = h_k(\mathbf{j})$ and $(-1)^{a_k(\mathbf{i})} = (-1)^{a_k(\mathbf{j})}$ for $2f+1 \leq k \leq n$. Hence we only need to check (3.20) when $\mathbf{i} \in P_{k,+}^n$, as all the other relations depend on $h_k(\mathbf{i})$, $(-1)^{a_k(\mathbf{i})}$, i_{k-1} , i_k and i_{k+1} .

First we prove the following equality holds: for any $2f+1 \leq k \leq n-1$, we have

$$\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \left(\sum_{\substack{1 \leq r \leq 2f \\ r \in A_{k,1}^{\mathbf{i}}}} y_r - 2 \sum_{\substack{1 \leq r \leq 2f \\ r \in A_{k,2}^{\mathbf{i}}}} y_r + \sum_{\substack{1 \leq r \leq 2f \\ r \in A_{k,3}^{\mathbf{i}}}} y_r - 2 \sum_{\substack{1 \leq r \leq 2f \\ r \in A_{k,4}^{\mathbf{i}}}} y_r \right) = 0. \quad (4.13)$$

Because we have $i_1 = i_3 = \dots = i_{2f-1} = -i_2 = -i_4 = \dots = -i_{2f} = \frac{\delta-1}{2}$, for any $1 \leq \ell \leq 2f$, $2\ell-1 \in A_{k,1}^{\mathbf{i}}$ if and only if $2\ell \in A_{k,3}^{\mathbf{i}}$, and $2\ell \in A_{k,1}^{\mathbf{i}}$ if and only if $2\ell-1 \in A_{k,3}^{\mathbf{i}}$. Similarly, $2\ell-1 \in A_{k,2}^{\mathbf{i}}$ if and only if $2\ell \in A_{k,4}^{\mathbf{i}}$, and $2\ell \in A_{k,2}^{\mathbf{i}}$ if and only if $2\ell-1 \in A_{k,4}^{\mathbf{i}}$. Hence by (3.24), (4.13) holds.

Suppose $\mathbf{i} \in P_{k,+}^n$ and $2f+1 \leq k \leq n-1$. Recall $(-1)^{a_k(\mathbf{i})} = (-1)^{a_k(\mathbf{j})}$. By (3.20) and (4.13) we have

$$\begin{aligned} \phi_{f,n}(\epsilon_{1,f} \epsilon_k e(\mathbf{i}) \epsilon_k) &= (-1)^{a_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}) \phi_{f,n}(\epsilon_{1,f} \left(\sum_{r \in A_{k,1}^{\mathbf{i}}} y_r - 2 \sum_{r \in A_{k,2}^{\mathbf{i}}} y_r + \sum_{r \in A_{k,3}^{\mathbf{i}}} y_r - 2 \sum_{r \in A_{k,4}^{\mathbf{i}}} y_r \right) \epsilon_k) \\ &= (-1)^{a_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}) \phi_{f,n}(\epsilon_{1,f} \left(\sum_{\substack{2f+1 \leq r \leq k-1 \\ r \in A_{k,1}^{\mathbf{i}}}} y_r - 2 \sum_{\substack{2f+1 \leq r \leq k-1 \\ r \in A_{k,2}^{\mathbf{i}}}} y_r + \sum_{\substack{2f+1 \leq r \leq k-1 \\ r \in A_{k,3}^{\mathbf{i}}}} y_r - 2 \sum_{\substack{2f+1 \leq r \leq k-1 \\ r \in A_{k,4}^{\mathbf{i}}}} y_r \right) \epsilon_k) \\ &= (-1)^{a_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}) \left(\sum_{r \in A_{k-2f,1}^{\mathbf{i}}} y_r - 2 \sum_{r \in A_{k-2f,2}^{\mathbf{i}}} y_r + \sum_{r \in A_{k-2f,3}^{\mathbf{i}}} y_r - 2 \sum_{r \in A_{k-2f,4}^{\mathbf{i}}} y_r \right) \epsilon_{k-2f} \\ &= \epsilon_{k-2f} e(\mathbf{j}) \epsilon_{k-2f}, \end{aligned}$$

which proves the Lemma. \square

The next result is used to prove Lemma 4.45.

4.44. Lemma. Suppose $\mathbf{i} = (i_1, \dots, i_n) \in P_{f,n}$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $t_{2f+1} = (1)$.

Proof. Suppose $\frac{\delta-1}{2} \neq \pm \frac{1}{2}$, the Lemma is obvious by the construction of \mathbf{t} . If $\frac{\delta-1}{2} = \frac{1}{2}$, we prove by induction.

When $f = 0$, the Lemma follows obviously. Suppose when $f < f'$ the Lemma follows. When $f = f'$, by induction we have $t_{2f-1} = (1)$. Hence by the construction, $\mathbf{t}(2f) = (2, 1)$ or $\mathbf{t}(2f) = -(1, 1)$. If $\mathbf{t}(2f) = -(1, 1)$, then $\mathbf{t}(2f+1) = (1, 1)$ and the Lemma holds. If $\mathbf{t}(2f) = (2, 1)$, then $\mathbf{t}(2f+1) = -(2, 1)$ and the Lemma holds. Hence when $\frac{\delta-1}{2} = \frac{1}{2}$, the Lemma holds. Following the same argument, the Lemma holds when $\frac{\delta-1}{2} = -\frac{1}{2}$. \square

4.45. Lemma. The map $\phi_{f,n}$ is a bijection.

Proof. By the definition, $\phi_{f,n}$ is surjective. In order to prove $\phi_{f,n}$ is injective, it suffices to show that $\ker \phi_{f,n} = \{0\}$ by checking the relations of $\mathcal{G}_{n-2f}(\delta)$. Following the same argument as in the proof of Lemma 4.43, we only need to check the first two relations of (3.8), and (3.20) when $\mathbf{i} \in P_{k,+}^n$.

Suppose $\mathbf{i} = (i_1, \dots, i_n) \in P^n$. For the first relation of (3.8), we prove that $e(\mathbf{i}) \epsilon_{1,f} e(\mathbf{i}) = 0$ when $i_{2f+1} \neq \frac{\delta-1}{2}$ and $e(\mathbf{i}) \epsilon_{1,f} y_{2f+1} e(\mathbf{i}) = 0$. Note that we have $c = \pm 1$ such that $e(\mathbf{i}) \epsilon_{1,f} e(\mathbf{i}) = c e(\mathbf{i}) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{i})$. Hence it suffices to prove that $e(\mathbf{i}) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{i}) = 0$ and $e(\mathbf{i}) \epsilon_1 \dots \epsilon_{2f-1} y_{2f+1} e(\mathbf{i}) = 0$ under certain conditions.

First we prove $e(\mathbf{i})\epsilon_1 \dots \epsilon_{2f-1}e(\mathbf{i}) = 0$ when $i_{2f+1} \neq \frac{\delta-1}{2}$. We apply the induction on f . When $f = 0$ it is obvious by (3.8). Suppose for $f - 1$ the result holds. By (3.24) we have

$$\begin{aligned} e(\mathbf{i})\epsilon_1 \dots \epsilon_{2f-1}e(\mathbf{i}) &= e(\mathbf{i})\epsilon_1 \dots \epsilon_{2f-3}\epsilon_{2f-1}\epsilon_{2f}e(\mathbf{l})\epsilon_{2f-1}e(\mathbf{i}) \\ &= e(\mathbf{i})\epsilon_{2f-1}\epsilon_{2f}e(\mathbf{l})\epsilon_1 \dots \epsilon_{2f-3}e(\mathbf{l})\epsilon_{2f-1}e(\mathbf{i}) \end{aligned}$$

where $\mathbf{l} \in P^n$. If we write $\mathbf{l} = (\ell_1, \ell_2, \dots, \ell_n)$, one can see that $\ell_{2f-1} = i_{2f+1} \neq \frac{\delta-1}{2}$ by (3.8), which implies $e(\mathbf{l})\epsilon_1 \dots \epsilon_{2f-3}e(\mathbf{l}) = 0$ by induction. Hence $e(\mathbf{i})\epsilon_1 \dots \epsilon_{2f-1}e(\mathbf{i}) = 0$ when $i_{2f+1} \neq \frac{\delta-1}{2}$.

Then we prove $e(\mathbf{i})\epsilon_1 \dots \epsilon_{2f-1}y_{2f+1}e(\mathbf{i}) = 0$. We apply induction on f as well. When $f = 0$ it is obvious by (3.8). Suppose for $f - 1$ the result holds. By (3.24) we have

$$\begin{aligned} e(\mathbf{i})\epsilon_1 \dots \epsilon_{2f-1}y_{2f+1}e(\mathbf{i}) &= e(\mathbf{i})\epsilon_1 \dots \epsilon_{2f-3}\epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f-1}y_{2f+1}e(\mathbf{i}) \\ &= e(\mathbf{i})\epsilon_{2f-1}\epsilon_{2f}\epsilon_1 \dots \epsilon_{2f-3}y_{2f-1}\epsilon_{2f-1}e(\mathbf{i}) = 0, \end{aligned}$$

where the last equality holds because $\epsilon_1 \dots \epsilon_{2f-3}y_{2f-1} = 0$ by induction. Hence the first relation of (3.8) consists.

For the second relation of (3.8), notice that $\epsilon_{1,f}e(\mathbf{i}) = 0$ if $e(\mathbf{i}) \notin P_{f,n}$ by (3.8). Hence, we have

$$\phi_{f,n}(\epsilon_{1,f} \cdot 1) = \phi_{f,n}(\epsilon_{1,f} \sum_{e(\mathbf{i}) \in P_{f,n}} e(\mathbf{i})) = \sum_{\mathbf{j} \in P^{n-2f}} e(\mathbf{j}) = 1.$$

For (3.20) when $\mathbf{i} \in P_{k,+}^{n-2f}$, the relation consists by following the same argument as in the proof of Lemma 4.43. \square

Lemma 4.45 shows that $\epsilon_{1,f}\mathcal{G}_{2f,n}(\delta) \cong \mathcal{G}_{n-2f}(\delta)$ as R -space. For $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$, define $R_{2f,n}(\delta)$ to be the subspace of $R_n(\delta)$ spanned by ψ_{st} 's, such that $\text{head}(\mathbf{s}), \text{head}(\mathbf{t}) \geq f$. By the definition of ψ_{st} 's, we have $R_{f,n}(\delta) \subseteq \epsilon_{1,f}\mathcal{G}_{2f,n}(\delta)$. Therefore, we restrict $\phi_{f,n}$ to $R_{f,n}(\delta)$.

4.46. Lemma. Suppose $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head}(\mathbf{s}), \text{head}(\mathbf{t}) \geq f$. Then we have $\phi_{f,n}(\psi_{st}) = \psi_{uv} \in R_{n-2f}(\delta)$. Moreover, if we write $\mathbf{s} = (\alpha_1, \dots, \alpha_n)$ and $\mathbf{t} = (\beta_1, \dots, \beta_n)$, then we have $\mathbf{u} = (\alpha_{2f+1}, \dots, \alpha_n)$ and $\mathbf{v} = (\beta_{2f+1}, \dots, \beta_n)$.

Proof. Because $\text{head}(\mathbf{s}), \text{head}(\mathbf{t}) \geq f$, we have

$$\begin{aligned} \alpha_1 &= \alpha_3 = \dots = \alpha_{2f-1} = -\alpha_2 = -\alpha_4 = \dots = -\alpha_{2f} = \alpha_0, \\ \beta_1 &= \beta_3 = \dots = \beta_{2f-1} = -\beta_2 = -\beta_4 = \dots = -\beta_{2f} = \alpha_0. \end{aligned}$$

Hence the Lemma follows by direct calculations. \square

By Lemma 4.46, one can see that $\phi_{f,n}(R_{f,n}(\delta)) = R_{n-2f}(\delta)$. Hence we have a sequence

$$R_n(\delta) = R_{0,n}(\delta) \supseteq R_{1,n}(\delta) \supseteq R_{2,n}(\delta) \supseteq \dots$$

Finally we introduce some application of restriction property of $R_n(\delta)$.

4.47. Lemma. Suppose $(\sigma, f) \in \mathcal{S}_n$. Then for any $(\lambda, f) \in \widehat{B}_{n-1}$ and $k \in P$, if $\text{res}(\alpha) \neq k$ for all $\alpha \in \mathcal{A}(\lambda)$, we have $\theta_k^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) = \sum c_{uv}\psi_{uv} \in R_n^{>f}(\delta)$, where $c_{uv} \neq 0$ only if $\text{head}(\mathbf{u}) \geq f$ and $\text{head}(\mathbf{v}) \geq f$.

Proof. If $f = 0$, the Lemma follows by Lemma 4.37. If $f > 0$, recall $\phi_{f,n}: \epsilon_{1,f}\mathcal{G}_{2f,n}(\delta) \rightarrow \mathcal{G}_{n-2f}(\delta)$ defined in Lemma 4.43. By the definition of $\theta_k^{(n-1)}$ and $\phi_{f,n}$ one can see that $\phi_{f,n} \circ \theta_k^{(n-1)} = \theta_k^{(n-2f-1)} \circ \phi_{f,n-1}$. By Lemma 4.45, $\phi_{f,n}$ is a bijection. Hence by Lemma 4.37 we have

$$\theta_k^{(n-1)}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}) = \phi_{f,n}^{-1}(\theta_k^{(n-2f-1)}(\phi_{f,n-1}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}))) = \phi_{f,n}^{-1}(\theta_k^{(n-2f-1)}(\psi_{\mathbf{t}(\lambda,0)\mathbf{t}(\lambda,0)})) = \sum_{\mathbf{s}, \mathbf{t}} c_{st}\phi_{f,n}^{-1}(\psi_{st}),$$

where $\mathbf{s}, \mathbf{t} \in \mathcal{T}_{n-2f}^{ud}(\mu)$ with $\mu \vdash n - 2f - 2m$ and $m > 0$. Hence the Lemma follows by Lemma 4.46. \square

4.48. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head}(\mathbf{t}) = h > 0$. Then for any $a \in \mathcal{G}_{2h,n}(\delta)$, we have

$$\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}a \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{\mathbf{t}(\lambda,f)\mathbf{v}} \pmod{R_n^{>f}(\delta)},$$

where $c_v \neq 0$ only if $\text{head}(\mathbf{v}) \geq h$.

Proof. Suppose $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$. Because $\text{head}(\mathbf{t}) = h \leq f$, we have $\alpha_1 = \alpha_3 = \dots = \alpha_{2h-1} = -\alpha_2 = -\alpha_4 = \dots = -\alpha_{2h} = \alpha_0$. Define $\mathbf{t} = (\alpha_{2h+1}, \alpha_{2h+2}, \dots, \alpha_n)$ and we have $\mathbf{t} \in \mathcal{T}_{n-2h}^{ud}(\lambda)$.

Recall $\phi_{h,n}: \epsilon_{1,h}\mathcal{G}_{2h,n}(\delta) \rightarrow \mathcal{G}_{n-2h}(\delta)$ defined in Lemma 4.43. Because $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}a \in \epsilon_{1,h}\mathcal{G}_{2h,n}(\delta)$, by Lemma 4.46, we have $\phi_{h,n}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}(\lambda,f)}a) = \psi_{\mathbf{t}(\lambda,f-h)\mathbf{t}(\lambda,f-h)}a'$, for some $a' \in \mathcal{G}_{n-2h}(\delta)$.

Because $(\lambda, f) \in \mathcal{S}_n$ and $h > 0$, we have $(\lambda, f - h) \in \widehat{B}_{n-2h} \subset \widehat{\mathcal{B}}$. Hence by the definition of $\widehat{\mathcal{B}}$, we have

$$\phi_{h,n}(\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}a) = \psi_{\mathbf{t}(\lambda,f-h)\mathbf{t}}a' \equiv \sum_{\check{\mathbf{v}} \in \mathcal{T}_{n-2h}^{ud}(\lambda)} c_{\check{\mathbf{v}}} \psi_{\mathbf{t}(\lambda,f-h)\check{\mathbf{v}}} \pmod{R_{n-2h}^{>f-h}(\delta)},$$

which implies $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}a \equiv \sum_{\check{\mathbf{v}} \in \mathcal{T}_{n-2h}^{ud}(\lambda)} c_{\check{\mathbf{v}}} \phi_{h,n}^{-1}(\psi_{\mathbf{t}(\lambda,f-h)\check{\mathbf{v}}}) \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathbf{v}} \psi_{\mathbf{t}(\lambda,f)\mathbf{v}} \pmod{R_n^{>f}(\delta)}$, where $c_{\mathbf{v}} \neq 0$ only if $\text{head}(\mathbf{v}) \geq h$. \square

5. A spanning set of $\mathcal{G}_n(\delta)$

In this section we prove $\{\psi_{\mathbf{st}} \mid \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda), (\lambda, f) \in \widehat{B}_n\}$ is a R -spanning set of $\mathcal{G}_n(\delta)$. The main idea is to prove $\bigcup_{i=1}^n \widehat{B}_i \subseteq \widehat{\mathcal{B}}$ by induction on \mathcal{S}_n . We show that if $(\lambda, f) \in \mathcal{S}_n$, we have

$$\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}a \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathbf{v}} \psi_{\mathbf{t}(\lambda,f)\mathbf{v}} \pmod{R_n^{>f}(\delta)}, \quad (5.1)$$

for any $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ and $a \in \mathcal{G}_n(\delta)$. As a byproduct, (5.1) shows the cellular-like property of $\psi_{\mathbf{st}}$'s, which will directly apply that $\{\psi_{\mathbf{st}} \mid \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda), (\lambda, f) \in \widehat{B}_n\}$ is a graded cellular basis of $\mathcal{G}_n(\delta)$ after we prove $\mathcal{G}_n(\delta) \cong \mathcal{B}_n(\delta)$.

5.1. The base case

Fix $(\lambda, f) \in \mathcal{S}_n$. We start by proving (5.1) in the most simple case: when $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f , which will be used in the following subsections for computational purposes when we prove more complicated cases. It suffices to prove (5.1) when a is one of the generators of $\mathcal{G}_n(\delta)$. For $e(\mathbf{i})$ with $\mathbf{i} \in P^n$ we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}e(\mathbf{i}) = \delta_{\mathbf{i},\mathbf{t}} \psi_{\mathbf{t}(\lambda,f)\mathbf{t}}$. Therefore it left us to consider y_k, ψ_s and ϵ_s with $1 \leq k \leq n$ and $1 \leq s \leq n-1$.

5.1. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . For $2f+1 \leq k \leq n-1$, we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k \in R_n^{>f}(\delta)$.

Proof. We have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}} = e_{(\lambda,f)}\psi_{d(\mathbf{t})}$ with $d(\mathbf{t}) \in \mathfrak{S}_{2f,n}$. Hence for any $\mathbf{k} \in P^n$, by Lemma 4.40 and Lemma 4.33 we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k e(\mathbf{k}) = e(\mathbf{i}_{(\lambda,f)})\psi_{d(\mathbf{t})}\epsilon_k e(\mathbf{k}) \in R_n^{>f}(\delta)$, which proves the Lemma. \square

5.2. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . Then for any $1 \leq k \leq n$, we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}y_k \in R_n^{>f}(\delta)$.

Proof. First we prove that for

$$\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}y_k = e_{(\lambda,f)}y_k \in R_n^{>f}(\delta), \quad (5.2)$$

the Lemma holds. When $f = 0$, the Lemma follows by Lemma 4.37 and Lemma 4.38. When $f \geq 1$, if $k = 1$, the Lemma follows by (3.8); and if $k = 2$, by (3.24) we have $e_{(\lambda,f)}y_2 = -e_{(\lambda,f)}y_1$, and the Lemma follows; and if $k \geq 3$ the Lemma follows by Lemma 4.48.

For arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f , suppose $d(\mathbf{t}) = s_{r_1}s_{r_2}\dots s_{r_\ell} \in \mathfrak{S}_{2f,n}$ is a reduced expression of $d(\mathbf{t})$. We prove the Lemma by induction. As the base step, when $\ell = 0$ the Lemma follows by (5.2). For the induction step, we assume that when $\ell < \ell'$ the Lemma holds. When $\ell = \ell'$, set $\mathbf{s} = \mathbf{t}^{(\lambda,f)}s_{r_1}\dots s_{r_{\ell-1}} \in \mathcal{T}_n^{ud}(\lambda)$ and $r_\ell = r$. One can see that $\mathbf{s} = \mathbf{t} \cdot s_r$. If we write $\mathbf{i}_t = (i_1, \dots, i_n)$, by Lemma 4.20 we have $|i_r - i_{r+1}| > 1$. Hence by (3.12) and (3.13), we have

$$\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}y_k = \psi_{\mathbf{t}(\lambda,f)\mathbf{s}}y_{s_r(k)}\psi_r \pm \psi_{\mathbf{t}(\lambda,f)\mathbf{s}}\epsilon_r e(\mathbf{i}_t) \in R_n^{>f}(\delta),$$

where the first term is in $R_n^{>f}(\delta)$ by induction, and the second term is in $R_n^{>f}(\delta)$ by Lemma 5.1. Hence we proves the Lemma. \square

5.3. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . Then for any $1 \leq k \leq n-1$, we have

$$\begin{cases} \psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\psi_k = \psi_{\mathbf{t}(\lambda,f)\mathbf{v}} & \text{if } \mathbf{v} = \mathbf{t} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda), \\ \psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\psi_k \in R_n^{>f}(\delta) & \text{if } \mathbf{t} \cdot s_k \text{ is not an up-down tableau.} \end{cases}$$

Proof. We prove the Lemma by considering consider the following different cases.

Case 1: $1 \leq k \leq 2f$.

In this case, as \mathbf{t} has head f , $\mathbf{t} \cdot s_k$ is not an up-down tableau. Write $\mathbf{i}_t = (i_1, i_2, \dots, i_n)$. Then we have $i_1 = i_3 = \dots = i_{2f-1} = \frac{\delta-1}{2}$ and $i_2 = i_4 = \dots = i_{2f} = -\frac{\delta-1}{2}$.

When $\frac{\delta-1}{2} = 0$, we have $\mathbf{i}_t \in P_{k,0}^n$. By (3.21) we have $e(\mathbf{i}_t)\psi_k e(\mathbf{i}_t) = e(\mathbf{i}_t)\psi_k e(\mathbf{i}_t)\epsilon_k e(\mathbf{i}_t) = 0$ and by (3.15) we have $e(\mathbf{i}_t)\epsilon_k e(\mathbf{i}_t) = (-1)^{a_k(\mathbf{i}_t)}e(\mathbf{i}_t)$. Hence, we have $e(\mathbf{i}_t)\psi_k e(\mathbf{i}_t) = e(\mathbf{i}_t)\psi_k = 0$, which implies $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\psi_k = 0$ by Lemma 4.10.

When $\frac{\delta-1}{2} \neq 0$, we have $h_k(\mathbf{i}_t) < 0$ by Corollary 3.5, which implies $h_k(\mathbf{i}_t \cdot s_k) > 0$. Hence $e(\mathbf{i}_t \cdot s_k) = 0$ by Lemma 4.41. Therefore by (3.9), we have $e(\mathbf{i}_t)\psi_k = \psi_k e(\mathbf{i}_t \cdot s_k) = 0$, which implies $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\psi_k = 0$ by Lemma 4.10. So the Lemma follows when $1 \leq k \leq 2f$.

Case 2: $2f+1 \leq k \leq n-1$ and $\mathbf{v} = \mathbf{t} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$.

In this case, as $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f , we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}} = e_{(\lambda,f)}\psi_{d(\mathbf{t})}$. By Lemma 4.11 and Lemma 4.15, we have $d(\mathbf{t}) \cdot s_k$ is semi-reduced correspond to $\mathbf{t}^{(\lambda,f)}$. Hence by Lemma 4.12, we have $e(\mathbf{i}_{(\lambda,f)})\psi_{d(\mathbf{t})}\psi_k = e(\mathbf{i}_{(\lambda,f)})\psi_{d(\mathbf{t}) \cdot s_k}$. Furthermore, as $\mathbf{v} = \mathbf{t} \cdot s_k$, we have $d(\mathbf{v}) = d(\mathbf{t}) \cdot s_k$, which implies $\psi_{d(\mathbf{t}) \cdot s_k} = \psi_{d(\mathbf{v})}$. Therefore, $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\psi_k = \psi_{\mathbf{t}(\lambda,f)\mathbf{v}}$, where the Lemma holds.

Case 3: $2f + 1 \leq k \leq n - 1$ and $\mathbf{t} \cdot s_k$ is not an up-down tableau.

Write $\mu = \mathbf{t}_{k-1}$. As $\mathbf{t} \cdot s_k$ is not an up-down tableau, we have $\text{res}(\alpha) \neq i_{k+1}$ for all $\alpha \in \mathcal{A}(\mu)$. Therefore, $e(\mathbf{i}_{\mathbf{t}} \cdot s_k)\epsilon_1 \dots \epsilon_{2f-1}e(\mathbf{i}_{\mathbf{t}} \cdot s_k) \in R_k^{>f}(\delta)$ by Lemma 4.37 and Lemma 4.38. Because we have

$$\begin{aligned}\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\psi_k &= e(\mathbf{i}_{(\lambda,f)})\epsilon_1 \dots \epsilon_{2f-1}e(\mathbf{i}_{(\lambda,f)})\psi_{d(\mathbf{t})}\psi_k \\ &= e(\mathbf{i}_{(\lambda,f)})\psi_{d(\mathbf{t})}\psi_k e(\mathbf{i}_{\mathbf{t}} \cdot s_k)\epsilon_1 \dots \epsilon_{2f-1}e(\mathbf{i}_{\mathbf{t}} \cdot s_k),\end{aligned}$$

the Lemma follows by Lemma 4.33. \square

5.4. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head f . Then for any $1 \leq k \leq n - 1$, we have

$$\begin{cases} \psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathbf{v}}\psi_{\mathbf{t}(\lambda,f)\mathbf{v}} \pmod{R_n^{>f}(\delta)}, & \text{if } 1 \leq k \leq 2f, \\ \psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k \in R_n^{>f}, & \text{if } 2f + 1 \leq k \leq n - 1. \end{cases}$$

Proof. Write $\mathbf{i}_{(\lambda,f)} = (i_1, \dots, i_n)$. We consider different values of k .

Case 1: $k = 2\ell - 1$ for $1 \leq \ell \leq f$.

We have $h_k(\mathbf{i}_{(\lambda,f)}) = -1$. Because $\psi_{d(\mathbf{t})} \in \mathfrak{S}_{2f,n}$, by (3.20), we have

$$\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_{2\ell-1} = e_{(\lambda,f)}\epsilon_{2\ell-1}\psi_{d(\mathbf{t})} = \begin{cases} (-1)^{a_{2\ell-1}(\mathbf{i}_{(\lambda,f)})}\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}, & \text{if } i_{2\ell-1} \neq -\frac{1}{2}, \\ 2(-1)^{a_{2\ell-1}(\mathbf{i}_{(\lambda,f)})}\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}f(y_1, \dots, y_{2\ell-2}), & \text{if } i_{2\ell-1} = -\frac{1}{2}, \end{cases}$$

where $f(y_1, \dots, y_{2\ell-2})$ is a polynomial of $y_1, \dots, y_{2\ell-2}$. Hence, by Lemma 5.2, the Lemma holds when $k = 2\ell - 1$ with $1 \leq \ell \leq f$.

Case 2: $k = 2\ell$ with $1 \leq \ell \leq f$.

Write $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$. Let $\beta = (2, 1)$ and $\gamma = (1, 2)$, and define

$$\begin{aligned}\mathbf{u} &= (\alpha_1, \dots, \alpha_{2\ell-1}, \beta, -\beta, \alpha_{2\ell+2}, \dots, \alpha_n), \\ \mathbf{v} &= (\alpha_1, \dots, \alpha_{2\ell-1}, \gamma, -\gamma, \alpha_{2\ell+2}, \dots, \alpha_n).\end{aligned}$$

If $\frac{\delta-1}{2} = \pm\frac{1}{2}$, we have $\mathbf{i}_{\mathbf{u}} = \mathbf{i}_{\mathbf{t}}$ or $\mathbf{i}_{\mathbf{v}} = \mathbf{i}_{\mathbf{t}}$. Hence, by (3.8), we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k = \psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k(e(\mathbf{i}_{\mathbf{u}}) + e(\mathbf{i}_{\mathbf{v}}))$. By directly comparing both sides of the equation, we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k = \psi_{\mathbf{t}(\lambda,f)\mathbf{u}} + \psi_{\mathbf{t}(\lambda,f)\mathbf{v}}$, because $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k e(\mathbf{i}_{\mathbf{u}}) = \psi_{\mathbf{t}(\lambda,f)\mathbf{u}}$ and $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k e(\mathbf{i}_{\mathbf{v}}) = \psi_{\mathbf{t}(\lambda,f)\mathbf{v}}$.

If $\frac{\delta-1}{2} \neq \pm\frac{1}{2}$, we have $\mathbf{i}_{\mathbf{t}} \neq \mathbf{i}_{\mathbf{u}}, \mathbf{i}_{\mathbf{v}}$. Hence, by (3.8), we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k = \psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k(e(\mathbf{i}_{\mathbf{t}}) + e(\mathbf{i}_{\mathbf{u}}) + e(\mathbf{i}_{\mathbf{v}}))$. Because $h_k(\mathbf{i}_{\mathbf{t}}) = -1$, by (3.15) we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k e(\mathbf{i}_{\mathbf{t}}) = \pm\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}$. Following the similar argument as above, we have $\psi_{\mathbf{t}(\lambda,f)\mathbf{t}}\epsilon_k = \pm\psi_{\mathbf{t}(\lambda,f)\mathbf{t}} + \psi_{\mathbf{t}(\lambda,f)\mathbf{u}} + \psi_{\mathbf{t}(\lambda,f)\mathbf{v}}$, which proves the Lemma when $1 \leq k \leq 2f$.

Case 3: $2f + 1 \leq k \leq n - 1$.

In this case the Lemma follows by Lemma 5.1. \square

5.2. A weak version

In this subsection we prove (5.1) for $a \in \mathcal{G}_{n-1}(\delta)$ by considering $\mathcal{G}_{n-1}(\delta)$ as a subalgebra of $\mathcal{G}_n(\delta)$. We separate the question into two cases by considering $\mathbf{t}(n) > 0$ and $\mathbf{t}(n) < 0$. For $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n) > 0$, (5.1) for $a \in \mathcal{G}_{n-1}(\delta)$ is directly implied by Lemma 4.42. Hence we have the following Lemma.

5.5. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n) > 0$. Then the equality (5.1) holds when $a \in \mathcal{G}_{n-1}(\delta)$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n) > 0$.

To prove (5.1) for $a \in \mathcal{G}_{n-1}(\delta)$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n) < 0$, first we introduce a commutation rule of $\mathcal{G}_n(\delta)$ as a technical result.

Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head $f - 1$ and $\mathbf{t}(n) < 0$. Let $\mathbf{s} = h(\mathbf{t}) \rightarrow \mathbf{t}$. It is easy to see that $\rho(\mathbf{s}, \mathbf{t}) = (a, n)$ because $\mathbf{t}(n) < 0$ and $\text{head}(\mathbf{t}) = f - 1$. The following Lemma gives the commutation rule of ψ_k and $\epsilon_{\mathbf{s} \rightarrow \mathbf{t}}$ when $2f + 2 \leq k \leq n - 1$.

5.6. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with head $f - 1$ and $\mathbf{t}(n) < 0$. Let $\mathbf{s} = h(\mathbf{t}) \rightarrow \mathbf{t}$ and $\rho(\mathbf{s}, \mathbf{t}) = (a, n)$. If $\mathbf{s} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$ for some $2f + 2 \leq k \leq n - 1$, there exists $w \in \mathfrak{S}_{2f-2, n-1}$ such that w is semi-reduced correspond to \mathbf{t} , and $\psi_k \epsilon_{\mathbf{s} \rightarrow \mathbf{t}} = \epsilon_{\mathbf{s} \cdot s_k \rightarrow \mathbf{t} \cdot w} \psi_{w^{-1}}$. In more details, we have

$$w = \begin{cases} s_{k-2}, & \text{if } 2f + 2 \leq k \leq a, \\ s_{k-1}s_{k-2}, & \text{if } k = a + 1, \\ s_{k-1}, & \text{if } a + 1 < k \leq n - 1. \end{cases}$$

Proof. As $\rho(\mathbf{s}, \mathbf{t}) = (a, n)$ and $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ with head f , we can write $\epsilon_{\mathbf{s} \rightarrow \mathbf{t}} = e(\mathbf{i}_{\mathbf{s}})\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}e(\mathbf{i}_{\mathbf{t}})$. Write $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$ and $\mathbf{s} = (\alpha_0, -\alpha_0, \alpha_1, \dots, \alpha_{a-1}, \alpha_{a+1}, \dots, \alpha_{n-1})$. We consider different values of k .

Case 1: $2f + 2 \leq k \leq a$.

By (3.38) we have

$$\psi_k \epsilon_{\mathbf{s} \rightarrow \mathbf{t}} = e(\mathbf{i}_{\mathbf{s}} \cdot s_k) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{k-2} \epsilon_{k-1} \epsilon_k \dots \epsilon_a \psi_{a+1} \dots \psi_{n-1} e(\mathbf{i}_{\mathbf{s}} \cdot s_{k-2}) \psi_{k-2}.$$

We write

$$\begin{aligned} \mathbf{s} \cdot s_k &= (\alpha_0, -\alpha_0, \alpha_1, \dots, \alpha_{k-3}, \alpha_{k-1}, \alpha_{k-2}, \alpha_k, \dots, \alpha_{a-1}, \alpha_{a+1}, \dots, \alpha_{n-1}), \\ \mathbf{t} \cdot s_{k-2} &= (\alpha_1, \dots, \alpha_{k-3}, \alpha_{k-1}, \alpha_{k-2}, \alpha_k, \dots, \alpha_n). \end{aligned}$$

As $\mathbf{s} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$, by Lemma 2.6, α_{k-2} and α_{k-1} are not adjacent, which implies $\mathbf{t} \cdot s_{k-2} \in \mathcal{T}_n^{ud}(\lambda)$ by Lemma 2.6. By the above expression, we have $\mathbf{s} \cdot s_k \rightarrow \mathbf{t} \cdot s_{k-2}$ and

$$\epsilon_{\mathbf{s} \cdot s_k \rightarrow \mathbf{t} \cdot s_{k-2}} = e(\mathbf{i}_{\mathbf{s}} \cdot s_k) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{n-1} e(\mathbf{i}_{\mathbf{t}} \cdot s_{k-2}).$$

Hence by setting $w = s_{k-2}$, the Lemma follows.

Case 2: $k = a + 1$.

By (3.40) we have

$$\psi_k \epsilon_{\mathbf{s} \rightarrow \mathbf{t}} = \psi_{a+1} \epsilon_{\mathbf{s} \rightarrow \mathbf{t}} = e(\mathbf{i}_{\mathbf{s}} \cdot s_{a+1}) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{a-1} \epsilon_a \epsilon_{a+1} \psi_{a+2} \dots \psi_{n-1} e(\mathbf{i}_{\mathbf{t}} \cdot s_a s_{a-1}) \psi_{a-1} \psi_a.$$

We write

$$\begin{aligned} \mathbf{s} \cdot s_k &= \mathbf{s} \cdot s_{a+1} = (\alpha_0, -\alpha_0, \alpha_1, \dots, \alpha_{a-2}, \alpha_{a+1}, \alpha_{a-1}, \alpha_{a+2}, \dots, \alpha_{n-1}), \\ \mathbf{t} \cdot s_{k-1} &= \mathbf{t} \cdot s_a = (\alpha_1, \dots, \alpha_{a-2}, \alpha_{a-1}, \alpha_{a+1}, \alpha_a, \alpha_{a+2}, \dots, \alpha_n), \\ \mathbf{t} \cdot s_{k-1} s_{k-2} &= \mathbf{t} \cdot s_a s_{a-1} = (\alpha_1, \dots, \alpha_{a-2}, \alpha_{a+1}, \alpha_{a-1}, \alpha_a, \alpha_{a+2}, \dots, \alpha_n). \end{aligned}$$

By Lemma 4.20, α_{a+1} is not adjacent to α_a , which implies $\mathbf{t} \cdot s_{k-1} \in \mathcal{T}_n^{ud}(\lambda)$ by Lemma 2.6. As $\mathbf{s} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$, by Lemma 2.6 we have α_{a-1} and α_{a+1} are not adjacent, which implies $\mathbf{t} \cdot s_{k-1} s_{k-2} \in \mathcal{T}_n^{ud}(\lambda)$ by Lemma 2.6. Hence $s_{k-1} s_{k-2}$ is semi-reduced correspond to \mathbf{t} . By the above expression, we have $\mathbf{s} \cdot s_k \rightarrow \mathbf{t} \cdot s_{k-1} s_{k-2}$ and

$$\epsilon_{\mathbf{s} \cdot s_k \rightarrow \mathbf{t} \cdot s_{k-1} s_{k-2}} = e(\mathbf{i}_{\mathbf{s}} \cdot s_{a+1}) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{a-1} \epsilon_a \epsilon_{a+1} \psi_{a+2} \dots \psi_{n-1} e(\mathbf{i}_{\mathbf{t}} \cdot s_a s_{a-1}).$$

By setting $w = s_{k-1} s_{k-2}$, the Lemma follows.

Case 3: $a + 1 < k \leq n - 1$.

Because ψ_k commutes with $\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1}$, we have

$$\psi_k \epsilon_{\mathbf{s} \rightarrow \mathbf{t}} = e(\mathbf{i}_{\mathbf{s}} \cdot s_k) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{k-2} \psi_k \psi_{k-1} \psi_k e(\mathbf{i}_{\mathbf{t}} \cdot s_{n-1} s_{n-2} \dots s_{k+1}) \psi_{k+1} \dots \psi_{n-1}.$$

If we write $\mathbf{i}_{\mathbf{t}} = (i_1, \dots, i_n)$, the relation of $\psi_k \psi_{k-1} \psi_k e(\mathbf{i}_{\mathbf{t}} \cdot s_{n-1} s_{n-2} \dots s_{k+1})$ is determined by i_{k-1} , i_k and i_n . Notice that $\alpha_r > 0$ and $i_r = \text{res}(\alpha_r)$ for $a \leq r \leq n - 1$, and $i_n = -\text{res}(\alpha_a)$.

Because $\alpha_n = -\alpha_a$, we have $\alpha_a \in \mathcal{R}(\mathbf{t}_{n-1})$. By Lemma 4.20, we have $\alpha_r > 0$ for all $a < r < n$. Hence, by the construction of up-down tableau we have $\text{res}(\alpha_r) \neq \text{res}(\alpha_a) = i_a = -i_n$ for any $a < r < n$. Therefore we have $i_{k-1} + i_n \neq 0$ and $i_k + i_n \neq 0$. Hence (3.25), (3.26), (3.29) and (3.30) will not apply here; if $i_{k-1} + i_k = 0$, by Lemma 4.20, α_{k-1} and α_k are not adjacent to α_a . α_k not adjacent to α_a implies $i_k \neq -i_n \pm 1$ and α_{k-1} not adjacent to α_a implies $i_k \neq -i_{k-1} \neq i_n \pm 1$. Hence (3.27) and (3.28) will not apply here; as $\mathbf{s} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$, by Lemma 2.6, α_{k-1} and α_k are not adjacent, which implies $|i_{k-1} - i_k| > 1$. Hence (3.31) and (3.32) will not apply here. Therefore we have $\psi_k \psi_{k-1} \psi_k e(\mathbf{i}_{\mathbf{t}} \cdot s_{n-1} s_{n-2} \dots s_{k+1}) = \psi_{k-1} \psi_k \psi_{k-1} e(\mathbf{i}_{\mathbf{t}} \cdot s_{n-1} s_{n-2} \dots s_{k+1})$, which implies

$$\psi_k \epsilon_{\mathbf{s} \rightarrow \mathbf{t}} = e(\mathbf{i}_{\mathbf{s}} \cdot s_k) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{k-2} \psi_{k-1} \psi_k \psi_{k+1} \dots \psi_{n-1} e(\mathbf{i}_{\mathbf{t}} \cdot s_{k-1}) \psi_{k-1}.$$

We write

$$\begin{aligned} \mathbf{s} \cdot s_k &= (\alpha_0, -\alpha_0, \alpha_1, \dots, \alpha_{a-1}, \alpha_{a+1}, \dots, \alpha_{k-2}, \alpha_k, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_{n-1}), \\ \mathbf{t} \cdot s_{k-1} &= (\alpha_1, \dots, \alpha_{k-2}, \alpha_k, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_n). \end{aligned}$$

As $\mathbf{s} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$, by Lemma 2.6, α_{k-1} and α_k are not adjacent, which implies $\mathbf{t} \cdot s_{k-1} \in \mathcal{T}_n^{ud}(\lambda)$ by Lemma 2.6. By the above expression, we have $\mathbf{s} \cdot s_k \rightarrow \mathbf{t} \cdot s_{k-1}$ and

$$\epsilon_{\mathbf{s} \cdot s_k \rightarrow \mathbf{t} \cdot s_{k-1}} = e(\mathbf{i}_{\mathbf{s}} \cdot s_k) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{k-2} \psi_{k-1} \psi_k \psi_{k+1} \dots \psi_{n-1} e(\mathbf{i}_{\mathbf{t}} \cdot s_{k-1})$$

By setting $w = s_{k-1}$, the Lemma follows. \square

Suppose \mathbf{s} and \mathbf{t} are defined as in Lemma 5.6. Let $\mu = \mathbf{t}_{n-1}$ and $\mathbf{u} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{u}|_{n-1} = \mathbf{t}^{(\mu, f-1)}$. It is easy to see that $\mathbf{u}(n) = \mathbf{t}(n)$. We abuse the symbol and define $d(\mathbf{t}) \in \mathcal{S}_{2f-2, n-1}$ such that $\mathbf{t} = \mathbf{u} \cdot d(\mathbf{t})$. Moreover, by the definition of \mathbf{u} , we have $\mathbf{t}^{(\lambda, f)} \rightarrow \mathbf{u}$.

5.7. Example. Suppose $(\lambda, f) = ((2), 2)$ and $n = 6$. Hence we have $(\lambda, f) \in \widehat{B}_n$. Let

$$t = (\emptyset, \square, \emptyset, \square, \square, \square).$$

Then we have $t \in \mathcal{T}_n^{ud}(\lambda)$ with head $f - 1$ and $t(n) < 0$. We can find a unique $s \in \mathcal{T}_n^{ud}(\lambda)$ such that $s = h(t) \rightarrow t$, which is

$$s = (\emptyset, \square, \emptyset, \square, \emptyset, \square).$$

We have $\mu = t_{n-1} = (2, 1)$, and define

$$u = (\emptyset, \square, \emptyset, \square, \square, \square).$$

Then we have $u \in \mathcal{T}_n^{ud}(\lambda)$ where $u|_{n-1} = t^{(\mu, f-1)}$, and $u(n) = t(n)$. We have $d(t) = s_4 \in \mathfrak{S}_{2f-1, n-1}$ and $t = u \cdot d(t)$. Also, we have $t^{(\lambda, f)} \rightarrow u$.

The next Lemma is an extension of Lemma 5.6.

5.8. Lemma. Suppose s, t and u are defined as above. Then we have $\psi_{d(s)}\epsilon_{s \rightarrow t} = \epsilon_{t^{(\lambda, f)} \rightarrow u}\psi_{d(t)}$.

Proof. By Lemma 5.6, there exists $w \in \mathfrak{S}_{2f-2, n-1}$ such that w is semi-reduced correspond to t and $\psi_{d(s)}\epsilon_{s \rightarrow t} = \epsilon_{t^{(\lambda, f)} \rightarrow t \cdot w}\psi_{w^{-1}}$.

Write $u = (\alpha_1, \dots, \alpha_n)$ and $\rho(t^{(\lambda, f)}, u) = (a, n)$. Define $v = t \cdot w$. As $w \in \mathfrak{S}_{n-1}$, we have $v(n) = t(n) = u(n)$. Hence $t^{(\lambda, f)} \rightarrow v$ and we write $\rho(t^{(\lambda, f)}, v) = (b, n)$. By the construction of u and v , we have $b \geq a$.

Define $d(v)$ such that $v = u \cdot d(v)$. Because $b \geq a$, we have $d(v) = s_a s_{a+1} \dots s_{b-1}$. By Lemma 4.20, for any $a < r < n$, α_r is not adjacent to α_a and $\alpha_r > 0$. Hence by Lemma 2.6, for any $a \leq k \leq b-1$, $u \cdot s_a s_{a+1} \dots s_k \in \mathcal{T}_n^{ud}(\lambda)$. Therefore $d(v)$ is semi-reduced correspond to u . As w is semi-reduced correspond to t , w^{-1} is semi-reduced correspond to $v = t \cdot w$. By Lemma 4.11, $d(v)w^{-1}$ is semi-reduced correspond to u . Hence we have $e(\mathbf{i}_u)\psi_{d(v)}\psi_{w^{-1}} = e(\mathbf{i}_u)\psi_{d(t)}$ by Lemma 4.12 as $u \cdot d(v)w^{-1} = u \cdot d(t) = t$.

Because $d(v)$ is semi-reduced correspond to u , by Corollary 4.14 we have $e(\mathbf{i}_v) = \psi_{d(v)^{-1}}e(\mathbf{i}_u)\psi_{d(v)}$. Therefore by (3.34), we have

$$\begin{aligned} \epsilon_{t^{(\lambda, f)} \rightarrow v}\psi_{w^{-1}} &= e(\mathbf{i}_{(\lambda, f)})\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_b\psi_{b+1} \dots \psi_{n-1}e(\mathbf{i}_v)\psi_{w^{-1}} \\ &= e(\mathbf{i}_{(\lambda, f)})\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_b\psi_{b+1} \dots \psi_{n-1}\psi_{b-1}\psi_{b-2} \dots \psi_a e(\mathbf{i}_u)\psi_{d(v)}\psi_{w^{-1}} \\ &= e(\mathbf{i}_{(\lambda, f)})\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}e(\mathbf{i}_u)\psi_{d(v)}\psi_{w^{-1}} = e_{t^{(\lambda, f)} \rightarrow u}\psi_{d(t)}, \end{aligned}$$

which completes the proof. \square

The key point of Lemma 5.8 is it implies the next Lemma, which allow us to apply induction to prove (5.1) for $a \in \mathcal{G}_{n-1}(\delta)$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t(n) < 0$.

In the rest of this subsection we fix $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t(n) = -\alpha < 0$, $\mu = \lambda \cup \{\alpha\}$ and $u \in \mathcal{T}_n^{ud}(\lambda)$ such that $u|_{n-1} = t^{(\mu, f-1)}$ and $u(n) = t(n)$.

5.9. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$. Then

(1) we have

$$\psi_{t^{(\lambda, f)} \rightarrow t} = \psi_{t^{(\lambda, f)} \rightarrow u}\epsilon_2\epsilon_4 \dots \epsilon_{2f-2}\theta_k^{(n-1)}(\psi_{t^{(\mu, f-1)} \rightarrow \dot{t}}), \quad (5.3)$$

where $k = -\text{res}(\alpha) \in P$ and $\dot{t} = t|_{n-1} \in \mathcal{T}_{n-1}^{ud}(\mu)$.

(2) for any $a \in \mathcal{G}_{n-1}(\delta)$ we have

$$\psi_{t^{(\lambda, f)} \rightarrow t}a \equiv \sum_{v \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{t^{(\lambda, f)} \rightarrow v} + \sum_{\substack{\dot{x}, \dot{y} \in \mathcal{T}_{n-1}^{ud}(\gamma) \\ (\gamma, f) \in \widehat{B}_{n-1}}} c_{\dot{x}\dot{y}} \psi_{t^{(\lambda, f)} \rightarrow u}\epsilon_2\epsilon_4 \dots \epsilon_{2f-2}\theta_k^{(n-1)}(\psi_{\dot{x}\dot{y}}) \pmod{R_n^{>f}(\delta)}, \quad (5.4)$$

where $c_{\dot{x}\dot{y}} \neq 0$ only if $\mathbf{i}_{\dot{x}} = \mathbf{i}_{(\mu, f-1)}$.

(3) for any $a \in \mathcal{G}_{n-1}(\delta)$ we have

$$\psi_{t^{(\lambda, f)} \rightarrow t}a \equiv \sum_{v \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{t^{(\lambda, f)} \rightarrow v} + \sum_{\substack{x, y \in \mathcal{T}_n^{ud}(\sigma) \\ (\sigma, f) \in \widehat{B}_n}} c_{xy} \psi_{t^{(\lambda, f)} \rightarrow u}\epsilon_2\epsilon_4 \dots \epsilon_{2f-2}\psi_{xy} \pmod{R_n^{>f}(\delta)}, \quad (5.5)$$

where $c_{xy} \neq 0$ only if $x(n) = y(n) > 0$ and $\mathbf{i}_x = \mathbf{i}_u$.

Proof. (1). As $t(n) < 0$, in the standard reduction sequence of t

$$h(t) = t^{(m)} \rightarrow t^{(m-1)} \rightarrow \dots \rightarrow t^{(1)} \rightarrow t^{(0)} = t,$$

we have $t^{(m-1)}(n) = t(n)$. Because $\dot{t} = t|_{n-1}$, by the definition, the standard reduction sequence of \dot{t} is

$$h(\dot{t}) = \dot{t}^{(m-1)} \rightarrow \dot{t}^{(m-2)} \rightarrow \dots \rightarrow \dot{t}^{(1)} \rightarrow \dot{t}^{(0)} = \dot{t},$$

where $\mathbf{i}^{(i)} = \mathbf{i}^{(i)}|_{n-1}$ and $\rho(\mathbf{i}^{(i)}, \mathbf{i}^{(i-1)}) = \rho(\mathbf{i}^{(i)}, \mathbf{i}^{(i-1)})$ for $1 \leq i \leq m-1$. Hence we have

$$\theta_k^{(n-1)}(\epsilon_i) = \theta_k^{(n-1)}(\epsilon_{\mathbf{i}^{(m-1)} \rightarrow \mathbf{i}^{(m-2)}} \dots \epsilon_{\mathbf{i}^{(1)} \rightarrow \mathbf{i}^{(0)}}) = \epsilon_{\mathbf{i}^{(m-1)} \rightarrow \mathbf{i}^{(m-2)}} \dots \epsilon_{\mathbf{i}^{(1)} \rightarrow \mathbf{i}^{(0)}}. \quad (5.6)$$

By (3.24) and (3.8), we have

$$e_{(\lambda, f)} = e(\mathbf{i}_{(\lambda, f)}) \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} e(\mathbf{i}_{(\lambda, f)}) = e_{(\lambda, f)} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \epsilon_1 \epsilon_3 \dots \epsilon_{2f-3} e(\mathbf{i}_{(\lambda, f)}). \quad (5.7)$$

Because $h(\mathbf{i}) = \mathbf{i}^{(m-1)}|_{n-1}$ and $\mathbf{i}^{(m-1)}(n) = \mathbf{i}(n) = \mathbf{u}(n)$, we have $\mathbf{u} \cdot d(h(\mathbf{i})) = \mathbf{i}^{(m-1)}$. Hence, by Lemma 5.8, we have

$$\psi_{\mathbf{i}^{(\lambda, f)} \mathbf{i}} = e_{(\lambda, f)} \psi_{d(h(\mathbf{i}))} \epsilon_{h(\mathbf{i}) \rightarrow \mathbf{i}^{(m-1)}} \epsilon_{\mathbf{i}^{(m-1)} \rightarrow \mathbf{i}^{(m-2)}} \dots \epsilon_{\mathbf{i}^{(1)} \rightarrow \mathbf{i}^{(0)}} = e_{(\lambda, f)} \epsilon_{\mathbf{i}^{(\lambda, f)} \rightarrow \mathbf{u}} \psi_{d(h(\mathbf{i}))} \epsilon_{\mathbf{i}^{(m-1)} \rightarrow \mathbf{i}^{(m-2)}} \dots \epsilon_{\mathbf{i}^{(1)} \rightarrow \mathbf{i}^{(0)}}.$$

Finally, because $\epsilon_{\mathbf{i}^{(\lambda, f)} \rightarrow \mathbf{u}} \in \mathcal{G}_{2f, n}(\delta)$, which commutes with $\epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \epsilon_1 \epsilon_3 \dots \epsilon_{2f-3}$, by (5.6) and (5.7) we have

$$\begin{aligned} \psi_{\mathbf{i}^{(\lambda, f)} \mathbf{i}} &= e_{(\lambda, f)} \epsilon_{\mathbf{i}^{(\lambda, f)} \rightarrow \mathbf{u}} \psi_{d(h(\mathbf{i}))} \epsilon_{\mathbf{i}^{(m-1)} \rightarrow \mathbf{i}^{(m-2)}} \dots \epsilon_{\mathbf{i}^{(1)} \rightarrow \mathbf{i}^{(0)}} \\ &= e_{(\lambda, f)} \epsilon_{\mathbf{i}^{(\lambda, f)} \rightarrow \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} e(\mathbf{i}_{\mathbf{u}}) \epsilon_1 \epsilon_3 \dots \epsilon_{2f-3} e(\mathbf{i}_{\mathbf{u}}) \psi_{d(h(\mathbf{i}))} \theta_k^{(n-1)}(\epsilon_i) \\ &= \psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \theta_k^{(n-1)}(e_{(\mu, f-1)}) \theta_k^{(n-1)}(\psi_{d(h(\mathbf{i}))}) \theta_k^{(n-1)}(\epsilon_i) \\ &= \psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \theta_k^{(n-1)}(\psi_{\mathbf{i}^{(\mu, f-1)} \mathbf{i}}), \end{aligned}$$

where $k = -\text{res}(\alpha) \in P$, which proves part (1).

(2). Because $(\lambda, f) \in \mathcal{S}_n$, by the definition of \mathcal{S}_n we have

$$\psi_{\mathbf{i}^{(\mu, f-1)} \mathbf{i}} \equiv \sum_{\check{\mathbf{v}} \in \mathcal{T}_{n-1}^{ud}(\mu)} c_{\check{\mathbf{v}}} \psi_{\mathbf{i}^{(\mu, f-1)} \check{\mathbf{v}}} + \sum_{\substack{\check{\mathbf{x}}, \check{\mathbf{y}} \in \mathcal{T}_{n-1}^{ud}(\gamma) \\ (\gamma, f) \in \widehat{B}_{n-1}}} c_{\check{\mathbf{x}} \check{\mathbf{y}}} \psi_{\check{\mathbf{x}} \check{\mathbf{y}}} \pmod{R_{n-1}^{>f}(\delta)},$$

where $c_{\check{\mathbf{x}} \check{\mathbf{y}}} \neq 0$ only if $\mathbf{i}_{\check{\mathbf{x}}} = \mathbf{i}_{(\mu, f-1)}$. By Lemma 4.38 we have $\theta_k^{(n-1)}(R_{n-1}^{>f}(\delta)) \subseteq R_n^{>f}(\delta)$. Therefore by substituting the above equation into (5.3) we have

$$\psi_{\mathbf{i}^{(\lambda, f)} \mathbf{i}} \equiv \sum_{\check{\mathbf{v}} \in \mathcal{T}_{n-1}^{ud}(\mu)} c_{\check{\mathbf{v}}} \psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \theta_k^{(n-1)}(\psi_{\mathbf{i}^{(\mu, f-1)} \check{\mathbf{v}}}) + \sum_{\substack{\check{\mathbf{x}}, \check{\mathbf{y}} \in \mathcal{T}_{n-1}^{ud}(\gamma) \\ (\gamma, f) \in \widehat{B}_{n-1}}} c_{\check{\mathbf{x}} \check{\mathbf{y}}} \psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \theta_k^{(n-1)}(\psi_{\check{\mathbf{x}} \check{\mathbf{y}}}) \pmod{R_n^{>f}(\delta)},$$

where $c_{\check{\mathbf{x}} \check{\mathbf{y}}} \neq 0$ only if $\mathbf{i}_{\check{\mathbf{x}}} = \mathbf{i}_{(\mu, f-1)}$. By (5.3), the first term of the equality equals $\sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathbf{v}} \psi_{\mathbf{i}^{(\lambda, f)} \mathbf{v}}$, where $\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{v}|_{n-1} = \check{\mathbf{v}}$ and $c_{\mathbf{v}} = c_{\check{\mathbf{v}}}$, which proves part (2).

(3). For $(\gamma, f) \in \widehat{B}_{n-1}$ and $\check{\mathbf{x}}, \check{\mathbf{y}} \in \mathcal{T}_{n-1}^{ud}(\gamma)$, if $\text{res}(\beta) \neq k$ for any $\beta \in \mathcal{A}(\gamma)$, we have $\theta_k^{(n-1)}(\psi_{\check{\mathbf{x}} \check{\mathbf{y}}}) \in R_n^{>f}(\delta)$ by Lemma 4.37. Then by Lemma 4.33, we have

$$\psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \theta_k^{(n-1)}(\psi_{\check{\mathbf{x}} \check{\mathbf{y}}}) \in R_n^{>f}(\delta). \quad (5.8)$$

If there exists $\beta \in \mathcal{A}(\gamma)$ with $\text{res}(\beta) = k$, set $\sigma = \gamma \cup \{\beta\}$. By Lemma 4.36, we have

$$\theta_k^{(n-1)}(\psi_{\check{\mathbf{x}} \check{\mathbf{y}}}) = \psi_{\mathbf{x} \mathbf{y}}, \quad (5.9)$$

where $\mathbf{x}, \mathbf{y} \in \mathcal{T}_n^{ud}(\sigma)$ such that $\mathbf{x}|_{n-1} = \check{\mathbf{x}}$ and $\mathbf{y}|_{n-1} = \check{\mathbf{y}}$. This implies that $\mathbf{x}(n) = \mathbf{y}(n) = \beta > 0$ and $\mathbf{i}_{\mathbf{x}} = \mathbf{i}_{\check{\mathbf{x}}} \vee k = \mathbf{i}_{(\mu, f-1)} \vee k = \mathbf{i}_{\mathbf{u}}$. Hence we prove part (3) by substituting (5.8) and (5.9) into (5.4). \square

The following Lemmas simplify the expression of $\psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \psi_{\mathbf{x} \mathbf{y}}$ on the RHS of (5.5).

5.10. Lemma. Suppose $(\sigma, f) \in \widehat{B}_n$ and $\mathbf{x}, \mathbf{y} \in \mathcal{T}_n^{ud}(\sigma)$ with $\mathbf{i}_{\mathbf{x}} = \mathbf{i}_{\mathbf{u}}$ and $\mathbf{x}(n) = \mathbf{y}(n) > 0$. Then we have

$$\psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \psi_{\mathbf{x} \mathbf{y}} \equiv c_{\sigma} \psi_{\mathbf{i}^{(\lambda, f)} \mathbf{u}} \left(\psi_{b-1} \dots \psi_{a+1} \epsilon_a \dots \epsilon_{2f} \epsilon_{2f-1} \right) \psi_{\mathbf{x}}^* e(\mathbf{i}_{(\sigma, f)}) \psi_{\mathbf{y}} \epsilon_{\mathbf{y}} \pmod{R_n^{>f}(\delta)},$$

where $c_{\sigma} \in R$.

Proof. Recall $\mathbf{u} \in \mathcal{T}_n^{ud}(\lambda)$ with head $f-1$. Hence if we write $\mathbf{i}_{\mathbf{x}} = (i_1, \dots, i_n)$, $\mathbf{i}_{\mathbf{u}} = \mathbf{i}_{\mathbf{x}}$ implies $i_1 = i_3 = \dots = i_{2f-3} = -i_2 = -i_4 = \dots = -i_{2f-2} = \frac{\delta-1}{2}$. By Lemma 4.44, we have $\mathbf{x}_{2f-1} = (1)$. When $\frac{\delta-1}{2} \neq \pm \frac{1}{2}$, one can see that for $1 \leq \ell \leq f-1$, we have $\mathbf{x}_{2\ell-1} = (1)$ and $\mathbf{x}_{2\ell} = \emptyset$; and when $\frac{\delta-1}{2} = \frac{1}{2}$, for $1 \leq \ell \leq f-1$ we have $\mathbf{x}_{2\ell-1} = (1)$ and $\mathbf{x}_{2\ell} = \emptyset$ or $(1, 1)$; and when $\frac{\delta-1}{2} = -\frac{1}{2}$, for $1 \leq \ell \leq f-1$ we have $\mathbf{x}_{2\ell-1} = (1)$ and $\mathbf{x}_{2\ell} = \emptyset$ or (2) .

Because $\text{Shape}(\mathbf{x}|_{2f-1}) = ((1), f-1)$ and $\mathbf{x} \in \mathcal{T}_n^{ud}(\sigma)$ where $(\sigma, f) \in \widehat{B}_n$, by the construction of $\epsilon_{\mathbf{x}}$ we have

$$\epsilon_{\mathbf{x}} = e(\mathbf{i}_{h(\mathbf{x})}) g_1 g_2 \dots g_{f-1} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{b-1} e(\mathbf{i}_{\mathbf{x}}), \quad (5.10)$$

where $2f \leq a < b \leq n$, and $g_{\ell} = 1$ if $\mathbf{x}_{2\ell} = \emptyset$ and $g_{\ell} = \epsilon_{2\ell}$ if $\mathbf{x}_{2\ell} \neq \emptyset$ for $1 \leq \ell \leq f-1$. By (3.20) and (3.24), we have

$$\epsilon_{2\ell-1} \epsilon_{2\ell} g_{\ell} \epsilon_{2\ell-1} \dots \epsilon_3 \epsilon_1 = f_{\ell} \epsilon_1 \epsilon_3 \dots \epsilon_{2\ell-1}, \quad (5.11)$$

where $f_\ell \in R$ if $g_\ell = 1$ and f_ℓ is a polynomial of $y_1, y_2, \dots, y_{2\ell-1}, y_{2\ell+2}$'s if $g_\ell = \epsilon_{2\ell}$. As $\psi_u, \psi_x^* \in \mathcal{G}_{2f,n}(\delta)$, they commute with $\epsilon_1 \epsilon_3 \dots \epsilon_{2f-1}$; and as $\epsilon_u \in \mathcal{G}_{2f-1,n}(\delta)$ which is implied by $\text{head}(u) = f - 1$, it commutes with $\epsilon_1 \epsilon_3 \dots \epsilon_{2f-3}$. Therefore, by (5.10) and (5.11) we have

$$\begin{aligned} & \psi_{\mathfrak{t}(\lambda,f)u} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \psi_{xy} \\ &= e(\mathbf{i}(\lambda,f)) \epsilon_{2f-1} \psi_u \epsilon_u \epsilon_1 \epsilon_3 \dots \epsilon_{2f-3} \epsilon_x^* \epsilon_{2f-1} \epsilon_{2f-3} \dots \epsilon_3 \epsilon_1 \cdot \psi_x^* e(\mathbf{i}(\sigma,f)) \psi_y \epsilon_y \\ &= e(\mathbf{i}(\lambda,f)) \epsilon_{2f-1} \psi_u \epsilon_u \epsilon_1 \epsilon_3 \dots \epsilon_{2f-3} \psi_{b-1} \dots \psi_{a+1} \epsilon_a \dots \epsilon_{2f} \epsilon_{2f-1} g_{f-1} \dots g_2 g_1 \epsilon_{2f-3} \dots \epsilon_3 \epsilon_1 \cdot \psi_x^* e(\mathbf{i}(\sigma,f)) \psi_y \epsilon_y \\ &= e(\mathbf{i}(\lambda,f)) \epsilon_{2f-1} \psi_u \epsilon_u \psi_{b-1} \dots \psi_{a+1} \epsilon_a \dots \epsilon_{2f} \epsilon_{2f-1} \cdot (\epsilon_1 \epsilon_3 \dots \epsilon_{2f-3} g_{f-1} \dots g_2 g_1 \epsilon_{2f-3} \dots \epsilon_3 \epsilon_1) \cdot \psi_x^* e(\mathbf{i}(\sigma,f)) \psi_y \epsilon_y. \end{aligned}$$

By applying (5.11) on the term in the bracket recursively, from $\ell = 1$ to $f - 1$, we have

$$\begin{aligned} \psi_{\mathfrak{t}(\lambda,f)u} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \psi_{xy} &= \left(\prod_{\ell=1}^{f-1} f_\ell \right) e(\mathbf{i}(\lambda,f)) \epsilon_{2f-1} \psi_u \epsilon_u \psi_{b-1} \dots \psi_{a+1} \epsilon_a \dots \epsilon_{2f} \epsilon_{2f-1} \cdot \epsilon_1 \epsilon_3 \dots \epsilon_{2f-3} \cdot \psi_x^* e(\mathbf{i}(\sigma,f)) \psi_y \epsilon_y \\ &= \left(\prod_{\ell=1}^{f-1} f_\ell \right) \left(e(\mathbf{i}(\lambda,f)) \epsilon_1 \epsilon_3 \dots \epsilon_{2f-1} \psi_u \epsilon_u \right) (\psi_{b-1} \dots \psi_{a+1} \epsilon_a \dots \epsilon_{2f} \epsilon_{2f-1}) \psi_x^* e(\mathbf{i}(\sigma,f)) \psi_y \epsilon_y \\ &= \left(\prod_{\ell=1}^{f-1} f_\ell \right) \psi_{\mathfrak{t}(\lambda,f)u} (\psi_{b-1} \dots \psi_{a+1} \epsilon_a \dots \epsilon_{2f} \epsilon_{2f-1}) \psi_x^* e(\mathbf{i}(\sigma,f)) \psi_y \epsilon_y. \end{aligned}$$

Finally, by Lemma 5.2, we have $y_s \psi_{\mathfrak{t}(\lambda,f)u} \in R_n^{>f}(\delta)$ for any $1 \leq s \leq n$. As $f_\ell \in R$ if $g_\ell = 1$ and f_ℓ is a polynomial of $y_1, y_2, \dots, y_{2\ell-1}, y_{2\ell+2}$'s if $g_\ell = \epsilon_{2\ell}$, we have

$$\left(\prod_{\ell=1}^{f-1} f_\ell \right) \psi_{\mathfrak{t}(\lambda,f)u} = c_\sigma \psi_{\mathfrak{t}(\lambda,f)u}$$

for some $c_\sigma \in R$, which completes the proof of the Lemma. \square

The following two Lemmas are technical results which will be used for computational purposes.

5.11. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head } f - 1$. Let $\mathfrak{s} = h(\mathfrak{t}) \rightarrow \mathfrak{t}$ and $\rho(\mathfrak{s}, \mathfrak{t}) = (a, b)$ for some $2f \leq a < b \leq n$. Then for any k with $2f - 1 \leq k \leq n - 1$, we have

$$\psi_{\mathfrak{t}(\lambda,f)\mathfrak{t}} \epsilon_k \epsilon_{k-1} \dots \epsilon_{2f-1} \equiv c_w \cdot \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \psi_w f_w \pmod{R_n^{>f}(\delta)},$$

where $c_w \in R$, $w \in \mathfrak{S}_{2f,n}$ and f_w is a polynomial of y_1, \dots, y_{k+1} .

Proof. It suffices to prove that for any $\mathbf{k} \in P^n$, we have

$$\psi_{\mathfrak{t}(\lambda,f)\mathfrak{t}} \epsilon_k \epsilon_{k-1} \dots \epsilon_{2f-1} e(\mathbf{k}) \equiv c_w \cdot \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \psi_w f_w e(\mathbf{k}) \pmod{R_n^{>f}(\delta)},$$

where $c_w \in R$, $w \in \mathfrak{S}_{a,b+1}$ and f_w is a polynomial of y_1, \dots, y_{k+1} . In the proof we omit the $e(\mathbf{k})$, but readers have to remember that there is always some $e(\mathbf{k})$ on the right of each element.

Because $\rho(\mathfrak{s}, \mathfrak{t}) = (a, b)$, we can write $\psi_{\mathfrak{t}(\lambda,f)\mathfrak{t}} = \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{b-1} e(\mathbf{i}_t)$. Because $\text{head}(\mathfrak{s}) = f$, we have $d(\mathfrak{s}) \in \mathfrak{S}_{2f+1,n}$, which implies $\psi_{\mathfrak{s}}$ commutes with ϵ_{2f} . Therefore, by (3.24), we have

$$\psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \epsilon_{2f} \epsilon_{2f-1} = e(\mathbf{i}(\lambda,f)) \epsilon_1 \dots \epsilon_{2f-1} e(\mathbf{i}(\lambda,f)) \psi_{\mathfrak{s}} \epsilon_{2f} \epsilon_{2f-1} = e(\mathbf{i}(\lambda,f)) \epsilon_1 \dots \epsilon_{2f-1} \epsilon_{2f} \epsilon_{2f-1} e(\mathbf{i}(\lambda,f)) \psi_{\mathfrak{s}} = \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}}. \quad (5.12)$$

We consider the following cases for different values of k , a and b .

Case 1: $k < a - 1$.

By (3.39), we have

$$\psi_{\mathfrak{t}(\lambda,f)\mathfrak{t}} \epsilon_k = \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{b-1} e(\mathbf{i}_t) \epsilon_k = \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \epsilon_{k+2} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{b-1}.$$

By Lemma 5.1 we have $\psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \epsilon_{k+2} \in R_n^{>f}(\delta)$. Hence by Lemma 4.33, we have $\psi_{\mathfrak{t}(\lambda,f)\mathfrak{t}} \epsilon_k \epsilon_{k-1} \dots \epsilon_{2f-1} \in R_n^{>f}(\delta)$.

Case 2: $k = a - 1$.

By (3.24) we have

$$\begin{aligned} \psi_{\mathfrak{t}(\lambda,f)\mathfrak{t}} \epsilon_{a-1} \dots \epsilon_{2f-1} &= \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{b-1} e(\mathbf{i}_t) \epsilon_{a-1} \dots \epsilon_{2f-1} \\ &= \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \epsilon_{2f} \epsilon_{2f-1} \psi_{a+1} \dots \psi_{b-1}. \end{aligned}$$

Hence, by (5.12) we have $\psi_{\mathfrak{t}(\lambda,f)\mathfrak{t}} \epsilon_{a-1} \dots \epsilon_{2f-1} = \psi_{\mathfrak{t}(\lambda,f)\mathfrak{s}} \psi_{a+1} \dots \psi_{b-1}$, where the Lemma holds by setting $c_w = 1$, $w = s_{a+1} s_{a+2} \dots s_{b-1}$ and $f_w = 1$.

Case 3: $k = a$.

If $k = a < b - 1$, by (3.37) and (3.24) we have

$$\begin{aligned}\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_a\epsilon_{a-1}\dots\epsilon_{2f-1} &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\dots\psi_{b-1}e(\mathbf{i}_t)\epsilon_a\epsilon_{a-1}\dots\epsilon_{2f-1} \\ &= \pm\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{a-1}\epsilon_a\psi_{a+2}\dots\psi_{b-1}e(\mathbf{i}_t)\epsilon_{a-1}\dots\epsilon_{2f-1} \\ &= \pm\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f-1}\psi_{a+2}\dots\psi_{b-1}.\end{aligned}$$

Hence by (5.12) we have $\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_a\epsilon_{a-1}\dots\epsilon_{2f-1} = \pm\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\psi_{a+2}\dots\psi_{b-1}$, where the Lemma holds by setting $c_w = \pm 1$, $w = s_{a+2}\dots s_{b-1}$ and $f_w = 1$.

If $k = a = b - 1$, by (3.20), (3.41) and (3.24), we have

$$\begin{aligned}\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_k e(\mathbf{i}_t)\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{k-1}f_w(y_1, \dots, y_{k-1})\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{k-1}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1}f_w(y_1, \dots, y_{2f-2}, y_{2f+1}, \dots, y_{k+1}) \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f-1}f_w(y_1, \dots, y_{2f-2}, y_{2f+1}, \dots, y_{k+1}),\end{aligned}$$

where f_w is a polynomial of y_1, \dots, y_{k+1} . Hence by (5.12) we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} = \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}f_w(y_1, \dots, y_{2f-2}, y_{2f+1}, \dots, y_{k+1}),$$

where the Lemma holds by setting $c_w = 1$, $w = 1$.

Case 4: $k = a + 1$.

When $k = a + 1 = b$, by (3.24) we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a e(\mathbf{i}_t)\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} = \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\epsilon_{a+1}\epsilon_a\dots\epsilon_{2f-1} = \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f-1},$$

and the Lemma holds by (5.12).

When $k = a + 1 = b - 1$, by (3.21) we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} = \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}e(\mathbf{i}_t)\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} = \pm\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\epsilon_{a+1}\epsilon_a\dots\epsilon_{2f-1},$$

and following the same argument as when $k = b$, the Lemma follows.

When $k = a + 1 < b - 1$, we have $a < b - 2$. Hence by (3.23), (3.14), (3.24) and (3.41), we have

$$\begin{aligned}\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\dots\psi_{b-1}e(\mathbf{i}_t)\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\psi_{a+2}\epsilon_{a+1}\epsilon_a\dots\epsilon_{2f-1}\psi_{a+3}\dots\psi_{b-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}^2\epsilon_{a+2}\epsilon_{a+1}\epsilon_a\dots\epsilon_{2f-1}\psi_{a+3}\dots\psi_{b-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a f(y_{a+1}, y_{a+2})\epsilon_{a+2}\epsilon_{a+1}\epsilon_a\dots\epsilon_{2f-1}\psi_{a+3}\dots\psi_{b-1},\end{aligned}\tag{5.13}$$

where $f(y_{a+1}, y_{a+2}) = 0, 1$ or $\pm(y_{a+2} - y_{a+1})$. By Lemma 5.1 and Lemma 4.33, for any $f(y_{a+1}, y_{a+2})$ we have that (5.13) is an element of $R_n^{>f}(\delta)$.

Case 5: $k > a + 1$.

When $k > b$, ϵ_k commutes with $\epsilon_{s \rightarrow t}$. Hence by Lemma 5.1 we have $\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} \in R_n^{>f}(\delta)$ and the Lemma follows.

When $k = b$, by (3.34) and (3.24) we have

$$\begin{aligned}\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\dots\psi_{b-1}e(\mathbf{i}_t)\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\dots\psi_{b-1}\epsilon_b\epsilon_{b-1}\dots\epsilon_{2f-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_b\psi_{b-1}\dots\psi_{a+2}\epsilon_{a+1}\epsilon_a\dots\epsilon_{2f-1} \\ &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f-1}\psi_b\psi_{b-1}\dots\psi_{a+2}.\end{aligned}$$

Hence by (5.12) we have $\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} = \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\psi_b\psi_{b-1}\dots\psi_{a+2}$, and the Lemma follows by setting $c_w = 1$, $w = s_b s_{b-1} \dots s_{a+2}$ and $f_w = 1$.

When $k = b - 1$, by (3.21) we have

$$\begin{aligned}\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\dots\psi_{b-1}e(\mathbf{i}_t)\epsilon_k\epsilon_{k-1}\dots\epsilon_{2f-1} &= \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\dots\psi_{b-1}\epsilon_{b-1}\epsilon_{b-2}\dots\epsilon_{2f-1} \\ &= \pm\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{s}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_a\psi_{a+1}\dots\psi_{b-2}\epsilon_{b-1}\epsilon_{b-2}\dots\epsilon_{2f-1},\end{aligned}$$

and following the same argument as when $k = b$, the Lemma holds.

When $k < b - 1$, following the similar argument as in Case 4 when $k < b - 1$, the Lemma follows. \square

5.12. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ with $f \geq 1$ and $\alpha \in \mathcal{A}(\lambda)$. Define $\mu = \lambda \cup \{\alpha\}$ and $\mathbf{u} \in \mathcal{T}_n^{ud}(\lambda)$ such that $\mathbf{u}|_{n-1} = \mathbf{t}^{(\mu, f-1)}$ and $\mathbf{u}(n) = -\alpha$. For any $2f \leq k < \ell \leq n - 1$, we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{u}}\psi_{\ell-1}\psi_{\ell-2}\dots\psi_{k+1}\epsilon_k\dots\epsilon_{2f}\epsilon_{2f-1} \equiv \sum_{w \in \mathcal{G}_{2f,n}} c_w \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}}\psi_w f_w \pmod{R_n^{>f}(\delta)},$$

where $c_w \in R$, $w \in \mathfrak{S}_{2f,n}$ and f_w 's are polynomials of y_1, \dots, y_n .

Proof. Because $u \in \mathcal{T}_n^{ud}(\lambda)$ and $u|_{n-1} = \mathbf{t}^{(\mu, f-1)}$, we have $\mathbf{t}^{(\lambda, f)} \rightarrow u$ and $\rho(\mathbf{t}^{(\lambda, f)}, u) = (a, n)$. Hence we can write

$$\psi_{\mathbf{t}^{(\lambda, f)}u} = e(\mathbf{i}_{(\lambda, f)})\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}e(\mathbf{i}_{(\lambda, f)})\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}e(\mathbf{i}_u).$$

We consider different values of ℓ and a .

Case 1: $\ell < a$.

By (3.38) and (3.39), we have

$$\begin{aligned} & \psi_{\mathbf{t}^{(\lambda, f)}u}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \\ &= e(\mathbf{i}_{(\lambda, f)})\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \\ &= e(\mathbf{i}_{(\lambda, f)})\psi_{\ell+1}\psi_{\ell} \dots \psi_{k+3}\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \\ &= e(\mathbf{i}_{(\lambda, f)})\psi_{\ell+1}\psi_{\ell} \dots \psi_{k+3}\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\epsilon_{k+2}\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}\epsilon_{k-1} \dots \epsilon_{2f}\epsilon_{2f-1}. \end{aligned}$$

By Lemma 4.40, we have $\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\epsilon_{k+2} \in R_n^{>f}(\delta)$. Hence $\psi_{\mathbf{t}^{(\lambda, f)}u}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \in R_n^{>f}(\delta)$ by Lemma 4.33.

Case 2: $\ell = a$.

By (3.34) and (3.24) we have

$$\begin{aligned} & \psi_{\mathbf{t}^{(\lambda, f)}u}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \\ &= e(\mathbf{i}_{(\lambda, f)})\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \\ &= e(\mathbf{i}_{(\lambda, f)})\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_{k+1}\psi_{k+2} \dots \psi_{n-1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \\ &= e(\mathbf{i}_{(\lambda, f)})\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\psi_{k+2} \dots \psi_{n-1} = \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}^{(\lambda, f)}\psi_w}, \end{aligned}$$

where $w = s_{k+2}s_{k+3} \dots s_{n-1}$.

Case 3: $a < \ell < n-1$.

By the assumption, we have $\ell \leq n-2$ and $a \leq n-3$. Hence we have $u(n-1) > 0$, $u(n) < 0$ and $u(n-1)+u(n) \neq 0$. By Lemma 4.24, we have $u \cdot s_{n-1} \in \mathcal{T}_n^{ud}(\lambda)$. Set $v = u \cdot s_{n-1}$ and we have $\psi_{\mathbf{t}^{(\lambda, f)}u} = \psi_{\mathbf{t}^{(\lambda, f)}v}\psi_{n-1}$. Therefore, we have

$$\psi_{\mathbf{t}^{(\lambda, f)}u}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} = \psi_{\mathbf{t}^{(\lambda, f)}v}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1}\psi_{n-1}. \quad (5.14)$$

By the construction, v has head $f-1$ and $v(n) = u(n-1) > 0$. By Lemma 4.42 and Lemma 4.48, we have

$$\psi_{\mathbf{t}^{(\lambda, f)}v}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1} \equiv \sum_{y \in \mathcal{T}_n^{ud}(\lambda)} c_y \psi_{\mathbf{t}^{(\lambda, f)}y} \pmod{R_n^{>f}(\delta)}, \quad (5.15)$$

where $c_y \neq 0$ only if $\text{head}(y) \geq f-1$ and $y(n) = v(n) > 0$. Therefore, we have $h(y) \rightarrow y$ and $\rho(h(y), y) = (s, m)$ where $s < m \leq n-1$. Because $k < \ell \leq n-2$, we have $k+1 \leq n-2$. Hence by Lemma 5.11, we have

$$\psi_{\mathbf{t}^{(\lambda, f)}y}\epsilon_k\epsilon_{k-1} \dots \epsilon_{2f-1}\psi_{n-1} \equiv c_w \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}^{(\lambda, f)}\psi_w} \psi_w f_w \psi_{n-1} \pmod{R_n^{>f}(\delta)},$$

where $w \in \mathfrak{S}_{2f,n}$ and f_w is a polynomial of y_1, \dots, y_{n-2} . Hence ψ_{n-1} commutes with f_w . By Lemma 5.3 and Lemma 4.33, we have

$$\psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}^{(\lambda, f)}\psi_w}\psi_w \psi_{n-1} \begin{cases} = \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}^{(\lambda, f)}\psi_{w'}} \psi_{w'}, & \text{if } w' = w \cdot s_{n-1} \text{ is semi-reduced corresponding to } \mathbf{t}^{(\lambda, f)}, \\ \in R_n^{>f}(\delta), & \text{if } w' = w \cdot s_{n-1} \text{ is not semi-reduced corresponding to } \mathbf{t}^{(\lambda, f)}. \end{cases}$$

Therefore, by Lemma 4.33, we have

$$\psi_{\mathbf{t}^{(\lambda, f)}y}\epsilon_k\epsilon_{k-1} \dots \epsilon_{2f-1}\psi_{n-1} \equiv c_w \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}^{(\lambda, f)}\psi_w}\psi_w \psi_{n-1} f_w \equiv c_{w'} \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}^{(\lambda, f)}\psi_{w'}} \psi_{w'} f_w \pmod{R_n^{>f}(\delta)},$$

where $c_{w'} \in R$, $w' \in \mathfrak{S}_{2f,n}$ and $f_{w'}$ is a polynomial of y_1, \dots, y_{n-2} . The Lemma holds by substituting the above equality and (5.15) into (5.14).

Case 4: $\ell = n-1$.

If $a = n-1$, then we have $\ell = a$, which is proved in Case 2. Assume $a < n-1$. Following the same argument as in Case 3, we have $u(n-1) > 0$ and $u \cdot s_{n-1} \in \mathcal{T}_n^{ud}(\lambda)$. Denote $v = u \cdot s_{n-1}$. By (3.35) we have

$$\begin{aligned} \psi_{\mathbf{t}^{(\lambda, f)}u}\psi_{\ell-1}\psi_{\ell-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} &= \psi_{\mathbf{t}^{(\lambda, f)}v}\psi_{n-1}\psi_{n-2} \dots \psi_{k+1}\epsilon_k \dots \epsilon_{2f}\epsilon_{2f-1} \\ &= \psi_{\mathbf{t}^{(\lambda, f)}v}\psi_k\psi_{k+1} \dots \psi_{n-2}\epsilon_{n-1} \dots \epsilon_{2f}\epsilon_{2f-1}. \end{aligned}$$

Because $\text{head}(v) = f-1$, by Lemma 4.42 and Lemma 4.48, we have

$$\psi_{\mathbf{t}^{(\lambda, f)}v}\psi_k\psi_{k+1} \dots \psi_{n-2}\epsilon_{n-1} \dots \epsilon_{2f}\epsilon_{2f-1} \equiv \sum_{y \in \mathcal{T}_n^{ud}(\lambda)} c_y \psi_{\mathbf{t}^{(\lambda, f)}y}\epsilon_{n-1} \dots \epsilon_{2f}\epsilon_{2f-1} \pmod{R_n^{>f}(\delta)},$$

where $c_y \neq 0$ only if $\text{head}(y) \geq f - 1$. Therefore, by Lemma 5.11, we have

$$\psi_{\mathbf{t}^{(\lambda,f)}y} \epsilon_{n-1} \dots \epsilon_{2f} \epsilon_{2f-1} \equiv c_w \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}^{(\lambda,f)}} \psi_w f_w \pmod{R_n^{>f}(\delta)},$$

where $c_w \in R$, $w \in \mathfrak{S}_{2f,n}$ and f_w is a polynomial of y_1, \dots, y_n . By combining the above two equalities, the Lemma holds. \square

Now we are ready to prove (5.1) when $a \in \mathcal{G}_{n-1}(\delta)$.

5.13. Lemma. *Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. For any $a \in \mathcal{G}_{n-1}(\delta)$, the equality (5.1) holds.*

Proof. When $\mathbf{t}(n) > 0$, the Lemma follows by Lemma 5.5. Hence we only have to consider the case when $\mathbf{t}(n) < 0$.

Suppose $\mathbf{t}(n) < 0$. By Lemma 5.9, we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}a} \equiv \sum_{v \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{\mathbf{t}^{(\lambda,f)}v} + \sum_{\substack{x,y \in \mathcal{T}_n^{ud}(\sigma) \\ (\sigma,f) \in \bar{B}_n}} c_{xy} \psi_{\mathbf{t}^{(\lambda,f)}u} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \psi_{xy} \pmod{R_n^{>f}(\delta)}, \quad (5.16)$$

where $c_{xy} \neq 0$ only if $x(n) = y(n) > 0$ and $\mathbf{i}_x = \mathbf{i}_u$. For the second term of (5.16), by Lemma 5.10 and Lemma 5.12, we have

$$\begin{aligned} \psi_{\mathbf{t}^{(\lambda,f)}u} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \psi_{xy} &\equiv c_\sigma \psi_{\mathbf{t}^{(\lambda,f)}u} (\psi_{b-1} \dots \psi_{a+1} \epsilon_a \dots \epsilon_{2f} \epsilon_{2f-1}) \psi_x^* e(\mathbf{i}_{(\sigma,f)}) \psi_y \epsilon_y \\ &\equiv \sum_{w \in \mathfrak{S}_{2f,n}} c_w c_\sigma \cdot \psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}^{(\lambda,f)}} \psi_w f_w \psi_x^* e(\mathbf{i}_{(\sigma,f)}) \psi_y \epsilon_y \pmod{R_n^{>f}(\delta)}, \end{aligned} \quad (5.17)$$

where $c_\sigma, c_w \in R$, $w \in \mathfrak{S}_{2f,n}$ and f_w 's are polynomials of y_1, \dots, y_n . Hence, by Lemma 5.2 and Lemma 5.3, we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}^{(\lambda,f)}} \psi_w f_w \psi_x^* e(\mathbf{i}_{(\sigma,f)}) \psi_y \epsilon_y \equiv c \cdot \psi_{\mathbf{t}^{(\lambda,f)}w} \epsilon_y \pmod{R_n^{>f}(\delta)},$$

where $c \in R$ and $w \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head } f$. We note that $c \neq 0$ only if $w = \mathbf{t}^{(\lambda,f)} w d(h(x))^* d(h(y))$ and $f_w = 1$. Notice that $\mathbf{t}(n) < 0$ forces $f > 0$. Hence we have $\epsilon_y \in \mathcal{G}_{2,n}(\delta)$ unless $\epsilon_y = e(\mathbf{i}_y)$. Because $\text{head}(w) = \text{head}(\mathbf{t}^{(\lambda,f)}) = f \geq 1$, by Lemma 4.45 we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}^{(\lambda,f)}} \psi_w f_w \psi_x^* e(\mathbf{i}_{(\sigma,f)}) \psi_y \epsilon_y \equiv c \cdot \psi_{\mathbf{t}^{(\lambda,f)}w} \epsilon_y \equiv \sum_{v' \in \mathcal{T}_n^{ud}(\lambda)} c_{v'} \psi_{\mathbf{t}^{(\lambda,f)}v'} \pmod{R_n^{>f}(\delta)}, \quad (5.18)$$

where $c_{v'} \in R$. Substitute (5.18) into (5.17) implies

$$\psi_{\mathbf{t}^{(\lambda,f)}u} \epsilon_2 \epsilon_4 \dots \epsilon_{2f-2} \psi_{xy} \equiv \sum_{v' \in \mathcal{T}_n^{ud}(\lambda)} c_{v'} \psi_{\mathbf{t}^{(\lambda,f)}v'} \pmod{R_n^{>f}(\delta)}.$$

Finally, the Lemma follows by substituting the above equality into (5.16). \square

5.3. The spanning set of $\mathcal{G}_n(\delta)$

In the previous subsection we have proved that (5.1) holds for any $a \in \mathcal{G}_{n-1}(\delta)$. In this subsection we extend this result to arbitrary $a \in \mathcal{G}_n(\delta)$ by showing that (5.1) holds when $a \in \{y_n, \psi_{n-1}, \epsilon_{n-1}\}$. Then using (5.1) we prove that $\psi_{\mathbf{st}}$'s form a spanning set of $\mathcal{G}_n(\delta)$.

We start by proving the equality (5.1) holds when $a = y_n$.

5.14. Lemma. *Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n) < 0$ and $\text{head } f - 1$. Then the equality (5.1) holds when $a = y_n$.*

Proof. As $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n) < 0$ and $\text{head } f - 1$, write $\mathbf{s} = h(\mathbf{t}) \rightarrow \mathbf{t}$ and we have $\rho(\mathbf{s}, \mathbf{t}) = (a, n)$ for $2f \leq a < n$. When $a = n - 1$, we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}y_n} = e_{(\lambda,f)} \psi_{d(\mathbf{s})} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} e(\mathbf{i}_t) y_n = -e_{(\lambda,f)} \psi_{d(\mathbf{s})} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} e(\mathbf{i}_t) y_{n-1} = -\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}y_{n-1}},$$

by (3.24), and the Lemma holds by Lemma 5.13.

When $a < n - 1$, we have $\mathbf{t}(n - 1) > 0$, $\mathbf{t}(n) < 0$ and $\mathbf{t}(n - 1) + \mathbf{t}(n) \neq 0$ by Lemma 4.20. By Lemma 2.6 we have $u = \mathbf{t} \cdot s_{n-1} \in \mathcal{T}_n^{ud}(\lambda)$. Hence the constructions of \mathbf{t} and u , we have $\mathbf{s} \rightarrow u$ and $\epsilon_{\mathbf{s} \rightarrow u} = e(\mathbf{i}_s) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{n-2} e(\mathbf{i}_u)$.

Write $\mathbf{i}_t = (i_1, \dots, i_n)$. Because $\mathbf{t}(n - 1) > 0$, $\mathbf{t}(n) < 0$ and $\mathbf{t}(n - 1) + \mathbf{t}(n) \neq 0$, we have $i_{n-1} + i_n \neq 0$. If $i_{n-1} \neq i_n$, by (3.12) and Lemma 5.13 we have

$$\psi_{\mathbf{t}^{(\lambda,f)}\mathbf{t}y_n} = \psi_{\mathbf{t}^{(\lambda,f)}u} y_{n-1} \psi_{n-1} \equiv \sum_{v \in \mathcal{T}_n^{ud}(\lambda)} c_w \psi_{\mathbf{t}^{(\lambda,f)}w} \psi_{n-1} \pmod{R_n^{>f}(\delta)}, \quad (5.19)$$

with $c_w \in R$. Moreover, because $y_{n-1} \in \mathcal{G}_{n-1}(\delta)$ and $u(n) = \mathbf{t}(n - 1) > 0$, $c_w \neq 0$ only if $w(n) > 0$; and because $\mathbf{t}(n) > 0$ forces $f \geq 1$ and $y_{n-1} \in \mathcal{G}_{2(f-1),n}(\delta)$, by Lemma 4.48 we have $c_w \neq 0$ only if $\text{head}(w) \geq f - 1$.

For any $\mathbf{w} \in \mathcal{T}_n^{ud}(\lambda)$ with $c_{\mathbf{w}} \neq 0$ in (5.19), by Lemma 4.10 we have $\mathbf{i}_{\mathbf{w}} = \mathbf{i}_{\mathbf{u}}$. If $\mathbf{w}(n-1) < 0$, $\text{head}(\mathbf{w}) \geq f-1$ forces $\mathbf{w} = \mathbf{u}$, which implies $\psi_{\mathbf{t}(\lambda, f)\mathbf{w}}\psi_{n-1} = \psi_{\mathbf{t}(\lambda, f)\mathbf{t}}$; and if $\mathbf{w}(n-1) > 0$, we have $\epsilon_{\mathbf{w}} \in \mathcal{G}_{n-2}(\delta)$ because $\mathbf{w}(n) > 0$, which implies

$$\psi_{\mathbf{t}(\lambda, f)\mathbf{w}}\psi_{n-1} = \psi_{\mathbf{t}(\lambda, f)\mathbf{s}}\epsilon_{\mathbf{w}}\psi_{n-1} = \psi_{\mathbf{t}(\lambda, f)\mathbf{s}}\psi_{n-1}\epsilon_{\mathbf{w}} \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathbf{v}}\psi_{\mathbf{t}(\lambda, f)\mathbf{v}} \pmod{R_n^{>f}(\delta)},$$

by (3.11), Lemma 5.3 and Lemma 5.13. Hence the Lemma holds when $i_{n-1} \neq i_n$.

If $i_{n-1} = i_n$, by (3.12) we have $\psi_{\mathbf{t}(\lambda, f)\mathbf{t}}\mathbf{y}_n = \psi_{\mathbf{t}(\lambda, f)\mathbf{u}}\psi_{n-1}\mathbf{y}_n = \psi_{\mathbf{t}(\lambda, f)\mathbf{u}}\mathbf{y}_{n-1}\psi_{n-1} + \psi_{\mathbf{t}(\lambda, f)\mathbf{u}}$. Hence the Lemma holds when $i_{n-1} = i_n$ by following the same argument as when $i_{n-1} \neq i_n$. \square

5.15. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. Then the equality (5.1) holds when $a = \mathbf{y}_n$.

Proof. Suppose $\text{head}(\mathbf{t}) = h$. We have the standard reduction sequence of \mathbf{t} :

$$h(\mathbf{t}) = \mathbf{t}^{(f-h)} \rightarrow \mathbf{t}^{(f-h-1)} \rightarrow \dots \rightarrow \mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)} = \mathbf{t}.$$

If $\mathbf{t}(n) > 0$, we have $\epsilon_{\mathbf{t}} \in \mathcal{G}_{n-1}(\delta)$. Hence by (3.10) we have $\psi_{\mathbf{t}(\lambda, f)\mathbf{t}}\mathbf{y}_n = \psi_{\mathbf{t}(\lambda, f)h(\mathbf{t})}\mathbf{y}_n\epsilon_{\mathbf{t}}$. By Lemma 5.2 and Lemma 5.13, the Lemma holds.

If $\mathbf{t}(n) < 0$, denote $\mathbf{s} = \mathbf{t}^{(f-h-1)}$. By the construction, we have $\mathbf{s}(n) = \mathbf{t}(n) < 0$ and $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head} \mathbf{s} = f-1$. Because $\epsilon_{\mathbf{t}^{(f-h-1)} \rightarrow \mathbf{t}^{(f-h-2)}} \dots \epsilon_{\mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)}} \in \mathcal{G}_{n-1}(\delta)$, by (3.10), Lemma 5.14 and Lemma 5.13 we have

$$\psi_{\mathbf{t}(\lambda, f)\mathbf{t}}\mathbf{y}_n = \psi_{\mathbf{t}(\lambda, f)\mathbf{s}}\mathbf{y}_n\epsilon_{\mathbf{t}^{(f-h-1)} \rightarrow \mathbf{t}^{(f-h-2)}} \dots \epsilon_{\mathbf{t}^{(1)} \rightarrow \mathbf{t}^{(0)}} \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathbf{v}}\psi_{\mathbf{t}(\lambda, f)\mathbf{v}} \pmod{R_n^{>f}(\delta)},$$

which completes the proof. \square

Next we prove that when $a = \epsilon_{n-1}$, the equality (5.1) holds. We separate the question by considering different $\mathbf{t}(n-1)$ and $\mathbf{t}(n)$. The next Lemma shows (5.1) holds when $\mathbf{t}(n-1) > 0$ and $\mathbf{t}(n) > 0$.

5.16. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n-1) > 0$ and $\mathbf{t}(n) > 0$. Then the equality (5.1) holds when $a = \epsilon_{n-1}$.

Proof. In this case we have $\epsilon_{\mathbf{t}}, \psi_{\mathbf{t}} \in \mathcal{G}_{n-2}(\delta)$, which commute with ϵ_{n-1} . Hence we have

$$\psi_{\mathbf{t}(\lambda, f)\mathbf{t}}\epsilon_{n-1} = e_{(\lambda, f)}\psi_{\mathbf{t}}\epsilon_{\mathbf{t}}\epsilon_{n-1} = e(\mathbf{i}_{(\lambda, f)})\epsilon_1\epsilon_3 \dots \epsilon_{2f-1}\epsilon_{n-1}\psi_{\mathbf{t}}\epsilon_{\mathbf{t}} \in R_n^{>f}(\delta),$$

by Lemma 4.40 and Lemma 4.33. \square

Before proceeding further, we introduce a technical result Lemma 5.19, which will be used to prove (5.1) when at least one of $\mathbf{t}(n-1)$ and $\mathbf{t}(n)$ is negative for $a = \epsilon_{n-1}$, and $a = \psi_{n-1}$. The following two Lemmas will be used to prove Lemma 5.19.

5.17. Lemma. Suppose $(\lambda, f-1) \in \widehat{B}_{n-2}$ and $(\mu, f) \in \widehat{B}_n$. We have either (1) $\mu = \lambda$, or (2) $\mathbf{u}(n-1) > 0$ and $\mathbf{u}(n) > 0$ if there exist $\mathbf{s} \in \mathcal{T}_{n-2}^{ud}(\lambda)$ and $\mathbf{u} \in \mathcal{T}_n^{ud}(\mu)$ such that the following conditions are satisfied for some $i \in P$:

- (1) $\text{head}(\mathbf{s}) = \text{head}(\mathbf{u}) = f-1$.
- (2) $\mathbf{i}_{\mathbf{u}} = (\mathbf{i}_{\mathbf{s}} \vee i, -i)$.
- (3) $\text{res}(\alpha) \neq i$ for all $\alpha \in \mathcal{A}(\lambda)$.

Proof. Because $\mathbf{u} \in \mathcal{T}_n^{ud}(\lambda)$ with $(\lambda, f) \in \widehat{B}_n$ and $\text{head}(\mathbf{u}) = f-1$, there exists a unique k with $2f+1 \leq k \leq n$ such that $\mathbf{u}(k) < 0$. If $k \leq n-2$, we have $\mathbf{u}(n-1) > 0$ and $\mathbf{u}(n) > 0$.

Whenever $n-1 \leq k \leq n$, let $\mathbf{x} = \mathbf{u}|_{n-2}$. Then we have $\text{Shape}(\mathbf{x}) = (\gamma, f-1) \in \widehat{B}_{n-2}$ for some partition γ and $\text{head}(\mathbf{x}) = \text{head}(\mathbf{u}) = f-1$. Moreover, because $\mathbf{i}_{\mathbf{u}} = (\mathbf{i}_{\mathbf{s}} \vee i, -i)$, we have $\mathbf{i}_{\mathbf{x}} = \mathbf{i}_{\mathbf{s}}$. By Lemma 4.36 it forces $\mathbf{x} = \mathbf{s}$ and $\mathbf{u}_{n-2} = \gamma = \lambda$.

Because $\text{res}(\alpha) \neq i$ for all $\alpha \in \mathcal{A}(\lambda)$, it forces $|\mathcal{A}_{\mathcal{R}_\lambda}(i)| = 1$. Therefore, by (3.5) - (3.7) we have $h_k(\mathbf{i}_{\mathbf{u}}) = 0$ or -1 , which implies $\mathbf{u}(n-1) + \mathbf{u}(n) = 0$ by Lemma 3.11. Hence, we have $\mathbf{u}_{n-2} = \mu$, which yields $\mu = \lambda$ as we have shown in the last paragraph that $\mathbf{u}_{n-2} = \lambda$. \square

5.18. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{t}, \mathbf{x}, \mathbf{y} \in \mathcal{T}_n^{ud}(\lambda)$. If $\text{head}(\mathbf{x}) \geq f-1$, we have

$$e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}\epsilon_{2f-1}\epsilon_{2f} \dots \epsilon_{n-1}\psi_{\mathbf{x}\mathbf{y}} \equiv c \cdot \psi_{\mathbf{t}(\lambda, f)\mathbf{y}} \pmod{R_n^{>f}(\delta)},$$

for some $c \in R$.

Proof. Because $\mathbf{x} \in \mathcal{T}_n^{ud}(\lambda)$ with $(\lambda, f) \in \widehat{B}_n$ and $\text{head}(\mathbf{x}) \geq f-1$, we have $\text{head}(\mathbf{x}) = f-1$ or f . When $\text{head}(\mathbf{x}) = f$, by Lemma 5.4 we have $\epsilon_{n-1}\psi_{\mathbf{x}\mathbf{y}} \in R_n^{>f}(\delta)$ and the Lemma follows by Lemma 4.33.

When $\text{head}(\mathbf{x}) = f-1$, let $\mathbf{w} = h(\mathbf{x}) \rightarrow \mathbf{x}$. By Lemma 5.11, we have

$$e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}\epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}\psi_{\mathbf{x}\mathbf{y}} \equiv c_w e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}f_w\psi_w\psi_{\mathbf{w}\mathbf{y}} \pmod{R_n^{>f}(\delta)},$$

where $c_w \in R$, $w \in \mathfrak{S}_n$ and f_w is a polynomial of y_1, \dots, y_n . As $\text{head}(\mathbf{w}) = f$, by Lemma 5.3 and Lemma 5.2, we have

$$e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}\epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}\psi_{\mathbf{x}\mathbf{y}} \equiv c_w e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}f_w\psi_w\psi_{\mathbf{w}\mathbf{y}} \equiv c_w \psi_{\mathbf{u}\mathbf{y}} \pmod{R_n^{>f}(\delta)},$$

for some $\mathbf{u} \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head}(\mathbf{u}) = f$. By Lemma 4.10, we have $\mathbf{i}_{\mathbf{u}} = \mathbf{i}_{(\lambda, f)}$. As \mathbf{u} has head f , by Lemma 4.36, it forces $\mathbf{u} = \mathbf{t}^{(\lambda, f)}$. Hence the Lemma holds as $c_w \in R$. \square

5.19. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\text{head}(\mathbf{t}) = f$. Then we have

$$\psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1} \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{v}} \pmod{R_n^{>f}(\delta)},$$

where $c_v \in R$.

Proof. It suffices to prove the Lemma by showing

$$\psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}e(\mathbf{j}) \equiv \sum_{\mathbf{v} \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{v}} \pmod{R_n^{>f}(\delta)}, \quad (5.20)$$

for any $\mathbf{j} \in P^n$. Notice that if we write $\mathbf{i}_{\mathbf{t}} = (i_1, \dots, i_n)$, $e(\mathbf{i}_{\mathbf{t}})\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}e(\mathbf{j}) \neq 0$ only if $j_r = i_r$ for $1 \leq r \leq 2f-1$, $j_r = i_{r+2}$ for $2f \leq r \leq n-2$ and $j_{n-1} + j_n = 0$ by (3.8). In the rest of the proof, we assume \mathbf{j} has such property.

Suppose there exists $\alpha \in \mathcal{A}(\lambda)$ such that $\text{res}(\alpha) = j_{n-1}$. Write $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$ and define

$$\mathbf{u} = (\alpha_1, \alpha_2, \dots, \alpha_{2f-1}, \alpha_{2f+2}, \alpha_{2f+3}, \dots, \alpha_n, \alpha, -\alpha).$$

Hence we have $\mathbf{i}_{\mathbf{u}} = \mathbf{j}$. Moreover, we have $\mathbf{t} \rightarrow \mathbf{u}$ and $\epsilon_{\mathbf{t} \rightarrow \mathbf{u}} = e(\mathbf{i}_{\mathbf{t}})\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}e(\mathbf{j})$. Therefore, by the definition of $\psi_{\mathbf{st}}$'s we have $\psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}e(\mathbf{j}) = \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{u}}$, which proves that (5.20) holds.

It left us to show that (5.20) holds when $\text{res}(\alpha) \neq j_{n-1}$ for all $\alpha \in \mathcal{A}(\lambda)$. In this case, we have

$$\psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}e(\mathbf{j}) = e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}\epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}e(\mathbf{j})\epsilon_1\epsilon_3\dots\epsilon_{2f-3}e(\mathbf{j}). \quad (5.21)$$

Because $j_r = i_r$ for $1 \leq r \leq 2f-1$, we have

$$e(\mathbf{j})\epsilon_1\epsilon_3\dots\epsilon_{2f-3}e(\mathbf{j}) = \theta_{j_n}^{(n-1)} \circ \theta_{j_{n-1}}^{(n-2)} \circ \dots \circ \theta_{j_{2f-1}}^{(2f-2)} (\psi_{\mathbf{t}^{(0, f-1)}\mathbf{t}^{(0, f-1)}}).$$

By applying Lemma 4.36 and Lemma 4.47 recursively to the above equality, we have

$$e(\mathbf{j})\epsilon_1\epsilon_3\dots\epsilon_{2f-3}e(\mathbf{j}) \equiv \sum_{\substack{(\mu, f) \in \widehat{B}_n \\ \mathbf{x}, \mathbf{y} \in \mathcal{T}_n^{ud}(\mu)}} c_{\mathbf{x}\mathbf{y}} \psi_{\mathbf{x}\mathbf{y}} \pmod{R_n^{>f}(\delta)}, \quad (5.22)$$

where $c_{\mathbf{x}\mathbf{y}} \neq 0$ only if $\text{head}(\mathbf{x}) \geq f-1$ and $\text{head}(\mathbf{y}) \geq f-1$. Moreover, by Lemma 4.10, we have $\mathbf{i}_{\mathbf{x}} = \mathbf{i}_{\mathbf{y}} = \mathbf{j}$ if $c_{\mathbf{x}\mathbf{y}} \neq 0$. Hence, by Lemma 5.17, we have either $\mathbf{x}, \mathbf{y} \in \mathcal{T}_n^{ud}(\lambda)$ or $\mathbf{x}(n-1) > 0$ and $\mathbf{x}(n) > 0$ if $c_{\mathbf{x}\mathbf{y}} \neq 0$.

Substituting (5.22) into (5.21) yields

$$\psi_{\mathbf{t}^{(\lambda, f)}\mathbf{t}}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}e(\mathbf{j}) \equiv \sum_{\substack{(\mu, f) \in \widehat{B}_n \\ \mathbf{x}, \mathbf{y} \in \mathcal{T}_n^{ud}(\mu)}} c_{\mathbf{x}\mathbf{y}} e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}\epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}\psi_{\mathbf{x}\mathbf{y}} \pmod{R_n^{>f}(\delta)}. \quad (5.23)$$

If $\mathbf{x}(n-1) > 0$ and $\mathbf{x}(n) > 0$, by Lemma 5.16 and Lemma 4.33 we have

$$e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}\epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}\psi_{\mathbf{x}\mathbf{y}} \in R_n^{>f}(\delta). \quad (5.24)$$

If $\mathbf{x}, \mathbf{y} \in \mathcal{T}_n^{ud}(\lambda)$, recall $\text{head}(\mathbf{x}) \geq f-1$. Hence by Lemma 5.18, we have

$$e(\mathbf{i}_{(\lambda, f)})\psi_{d(\mathbf{t})}\epsilon_{2f-1}\epsilon_{2f}\epsilon_{2f+1}\dots\epsilon_{n-1}\psi_{\mathbf{x}\mathbf{y}} \equiv c \cdot \psi_{\mathbf{t}^{(\lambda, f)}\mathbf{y}} \pmod{R_n^{>f}(\delta)}, \quad (5.25)$$

where $c \in R$. Hence (5.20) follows by substituting (5.24) and (5.25) into (5.23). \square

Then we consider the cases when at least one of $\mathbf{t}(n-1)$ and $\mathbf{t}(n)$ is negative.

5.20. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{t}(n-1) < 0$ and $\mathbf{t}(n) < 0$. Then the equality (5.1) holds when $a = \epsilon_{n-1}$.

Proof. Because $t(n-1) < 0$ and $t(n) < 0$, the standard reduction sequence t is

$$s = t^{(m)} \rightarrow t^{(m-1)} \rightarrow \dots t^{(1)} \rightarrow t^{(0)} = t,$$

where $\rho(t^{(m)}, t^{(m-1)}) = (a, n)$ and $\rho(t^{(m-1)}, t^{(m-2)}) = (b, n-1)$ for $a \leq n-1$ and $b \leq n-2$. Hence we can write

$$\begin{aligned} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} &= e(\mathbf{i}_{t^{(m)}}) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{n-1} e(\mathbf{i}_{t^{(m-1)}}), \\ \epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} &= e(\mathbf{i}_{t^{(m-1)}}) \epsilon_{2f-2} \epsilon_{2f-1} \dots \epsilon_b \psi_{b+1} \dots \psi_{n-2} e(\mathbf{i}_{t^{(m-2)}}). \end{aligned}$$

By (3.34), we have

$$\epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} \epsilon_{n-1} = e(\mathbf{i}_{t^{(m-1)}}) \epsilon_{2f-2} \epsilon_{2f-1} \dots \epsilon_{n-1} (\psi_{n-3} \psi_{n-4} \dots \psi_b); \quad (5.26)$$

and by (3.38) and (3.39) we have

$$\epsilon_{t^{(m)} \rightarrow t^{(m-1)}} \epsilon_{2f-2} \epsilon_{2f-1} \dots \epsilon_{n-1} = e(\mathbf{i}_{t^{(m)}}) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} (\epsilon_{2f-2} \epsilon_{2f-1} \dots \epsilon_{a-2} \psi_{a-1} \dots \psi_{n-3}). \quad (5.27)$$

Set $x = \epsilon_{2f-2} \epsilon_{2f-1} \dots \epsilon_{a-2} \psi_{a-1} \dots \psi_{n-3} \psi_{n-3} \psi_{n-4} \dots \psi_b$ and $y = \epsilon_{t^{(m-2)} \rightarrow t^{(m-3)}} \dots \epsilon_{t^{(1)} \rightarrow t} \in \mathcal{G}_{n-2}(\delta)$. Because $a \leq n-1$ and $b \leq n-2$, we have $x \in \mathcal{G}_{n-1}(\delta)$. Then by (3.11), (5.26) and (5.27), we have

$$\begin{aligned} \psi_{t^{(\lambda, f)} t} \epsilon_{n-1} &= \psi_{t^{(\lambda, f)} s} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} \epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} y \epsilon_{n-1} = \psi_{t^{(\lambda, f)} s} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} \epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} \epsilon_{n-1} y \\ &= \psi_{t^{(\lambda, f)} s} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} \cdot xy. \end{aligned}$$

Because $s = h(t) \in \mathcal{T}_n^{ud}(\lambda)$, the head of s is f . Then because $xy \in \mathcal{G}_{n-1}(\delta)$, the Lemma follows by Lemma 5.19 and Lemma 5.13. \square

5.21. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t(n-1) < 0$, $t(n) > 0$ or $t(n-1) > 0$, $t(n) < 0$. Then the equality (5.1) holds when $a = \epsilon_{n-1}$.

Proof. Suppose $\mathbf{i}_t = (i_1, \dots, i_n)$. We assume $i_{n-1} + i_n = 0$, otherwise by (3.8) we have $\psi_{t^{(\lambda, f)} t} \epsilon_{n-1} = 0$. Hence, as $t(n-1) < 0$, $t(n) > 0$ or $t(n-1) > 0$, $t(n) < 0$, by the construction of up-down tableaux, we have $t(n-1) + t(n) = 0$.

Let the standard reduction sequence of t be

$$s = t^{(m)} \rightarrow t^{(m-1)} \rightarrow \dots t^{(1)} \rightarrow t^{(0)} = t,$$

and denote $x = \epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} \epsilon_{t^{(m-2)} \rightarrow t^{(m-3)}} \dots \epsilon_{t^{(1)} \rightarrow t}$. Notice that because $t(n-1) > 0$, $t(n) < 0$ or $t(n-1) < 0$, $t(n) > 0$, we have $x \in \mathcal{G}_{n-2}(\delta)$ in either cases. Hence by (3.11), x commutes with ϵ_{n-1} .

When $t(n-1) > 0$ and $t(n) < 0$, we have $\rho(t^{(m)}, t^{(m-1)}) = (n-1, n)$. Then by (3.20) and (3.41), we have

$$\begin{aligned} \psi_{t^{(\lambda, f)} t} \epsilon_{n-1} &= \psi_{t^{(\lambda, f)} s} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} x \epsilon_{n-1} = \psi_{t^{(\lambda, f)} s} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} \epsilon_{n-1} x \\ &= \psi_{t^{(\lambda, f)} s} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} e(\mathbf{i}_{t^{(m-1)}}) \epsilon_{n-1} x \\ &= \psi_{t^{(\lambda, f)} s} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} f(y_1, \dots, y_{n-2}) x, \end{aligned}$$

where $f(y_1, \dots, y_{n-2})$ is a polynomial of y_1, \dots, y_{n-2} .

Because $s = h(t) \in \mathcal{T}_n^{ud}(\lambda)$ with head f and $f(y_1, \dots, y_{n-2})x \in \mathcal{G}_{n-2}(\delta)$, by Lemma 5.19 and Lemma 5.13, the Lemma holds when $t(n-1) > 0$ and $t(n) < 0$.

When $t(n-1) < 0$ and $t(n) > 0$, we have $\rho(t^{(m)}, t^{(m-1)}) = (a, n-1)$ with $a \leq n-2$. Then by (3.34) we have

$$\begin{aligned} \psi_{t^{(\lambda, f)} t} \epsilon_{n-1} &= \psi_{t^{(\lambda, f)} s} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} x \epsilon_{n-1} = \psi_{t^{(\lambda, f)} s} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} \epsilon_{n-1} x \\ &= \psi_{t^{(\lambda, f)} s} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{n-2} \epsilon_{n-1} x \\ &= \psi_{t^{(\lambda, f)} s} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} \psi_{n-3} \psi_{n-4} \dots \psi_a x. \end{aligned}$$

Because $s = h(t) \in \mathcal{T}_n^{ud}(\lambda)$ with head f and $\psi_{n-3} \psi_{n-4} \dots \psi_a x \in \mathcal{G}_{n-2}(\delta)$, by Lemma 5.19 and Lemma 5.13, the Lemma holds when $t(n-1) < 0$ and $t(n) > 0$. \square

Combining Lemma 5.16, Lemma 5.20 and Lemma 5.21, the next Lemma follows.

5.22. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$. Then the equality (5.1) holds when $a = \epsilon_{n-1}$.

Finally, we prove that when $a = \psi_{n-1}$, the equality (5.1) holds. Similar as before, we separate the question by considering different $t(n-1)$ and $t(n)$.

5.23. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t(n) > 0$. Then the equality (5.1) holds when $a = \psi_{n-1}$.

Proof. When $t(n-1) > 0$, we have $\epsilon_t \in \mathcal{G}_{n-2}(\delta)$, which commutes with ψ_{n-1} . Denote $\mathbf{s} = h(t) \in \mathcal{T}_n^{ud}(\lambda)$, and we have $\text{head}(\mathbf{s}) = f$. By Lemma 5.3 and Lemma 5.13, we have

$$\psi_{t(\lambda, f)} \psi_{n-1} = \psi_{t(\lambda, f) \mathbf{s}} \psi_{n-1} \epsilon_t \equiv \sum_{v \in \mathcal{T}_n^{ud}(\lambda)} c_v \psi_{t(\lambda, f) v} \pmod{B_n^{>f}(\delta)}.$$

When $t(n-1) < 0$, write $t(n-1) = -\alpha$ and $t(n) = \beta$. If $\alpha \neq \beta$, by Lemma 2.6 we have $u = t \cdot s_{n-1} \in \mathcal{T}_n^{ud}(\lambda)$ and by the definition of ψ_{st} 's, we have $\psi_{t(\lambda, f)} \psi_{n-1} = \psi_{t(\lambda, f) u}$. Hence the Lemma holds.

If $\alpha = \beta$, let the standard reduction sequence of t be

$$\mathbf{s} = t^{(m)} \rightarrow t^{(m-1)} \rightarrow \dots \rightarrow t^{(1)} \rightarrow t^{(0)} = t,$$

and denote $x = \epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} \epsilon_{t^{(m-2)} \rightarrow t^{(m-3)}} \dots \epsilon_{t^{(1)} \rightarrow t}$. Notice that as $t(n-1) < 0$ and $t(n) > 0$, we have $x \in \mathcal{G}_{n-2}(\delta)$, which commutes with ψ_{n-1} . Let $\rho(t^{(m)}, t^{(m-1)}) = (a, n-1)$ with $a \leq n-2$. By (3.34) we have

$$\begin{aligned} \psi_{t(\lambda, f)} \psi_{n-1} &= \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} x \psi_{n-1} = \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} \psi_{n-1} x \\ &= \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{n-2} \psi_{n-1} x \\ &= \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} (\psi_{n-2} \psi_{n-3} \dots \psi_a x). \end{aligned}$$

Because $\mathbf{s} = h(t) \in \mathcal{T}_n^{ud}(\lambda)$ with head f and $\psi_{n-2} \psi_{n-3} \dots \psi_a x \in \mathcal{G}_{n-1}(\delta)$, the Lemma follows by Lemma 5.19 and Lemma 5.13. \square

5.24. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t(n-1) > 0$ and $t(n) < 0$. Then the equality (5.1) holds when $a = \psi_{n-1}$.

Proof. Let the standard reduction sequence of t be

$$\mathbf{s} = t^{(m)} \rightarrow t^{(m-1)} \rightarrow \dots \rightarrow t^{(1)} \rightarrow t^{(0)} = t,$$

and denote $x = \epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} \epsilon_{t^{(m-2)} \rightarrow t^{(m-3)}} \dots \epsilon_{t^{(1)} \rightarrow t}$. Notice that as $t(n-1) > 0$ and $t(n) < 0$, we have $x \in \mathcal{G}_{n-2}(\delta)$, which commutes with ψ_{n-1} .

Suppose $t(n-1) = \beta$ and $t(n) = -\alpha$. If $\alpha \neq \beta$, by Lemma 2.6 we have $u = t \cdot s_{n-1} \in \mathcal{T}_n^{ud}(\lambda)$ and $\psi_{t(\lambda, f)} t = \psi_{t(\lambda, f) u} \psi_{n-1}$ by the construction of $\psi_{t(\lambda, f) t}$ and $\psi_{t(\lambda, f) u}$. Hence by (3.14) we have

$$\psi_{t(\lambda, f)} \psi_{n-1} = \psi_{t(\lambda, f) u} \psi_{n-1}^2 = \psi_{t(\lambda, f) u} f(y_{n-1}, y_n),$$

where $f(y_{n-1}, y_n)$ is a polynomial of y_{n-1} and y_n determined by \mathbf{i}_u . Hence the Lemma holds by Lemma 5.15.

If $\alpha = \beta$, we have $\rho(t^{(m)}, t^{(m-1)}) = (n-1, n)$. By (3.22), we have

$$\begin{aligned} \psi_{t(\lambda, f)} \psi_{n-1} &= \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} x \psi_{n-1} = \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} \psi_{n-1} x \\ &= \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} e(\mathbf{i}_t) \psi_{n-1} x \\ &= c \cdot \psi_{t(\lambda, f) \mathbf{s}} \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_{n-1} x, \end{aligned}$$

where $c \in R$. Hence, as $\mathbf{s} = h(t) \in \mathcal{T}_n^{ud}(\lambda)$ with head f and $x \in \mathcal{G}_{n-2}(\delta)$, by Lemma 5.19 and Lemma 5.13, the Lemma holds. \square

5.25. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with $t(n-1) < 0$ and $t(n) < 0$. Then the equality (5.1) holds when $a = \psi_{n-1}$.

Proof. Let the standard reduction sequence of t be

$$\mathbf{s} = t^{(m)} \rightarrow t^{(m-1)} \rightarrow \dots \rightarrow t^{(1)} \rightarrow t^{(0)} = t,$$

and denote $x = \epsilon_{t^{(m-2)} \rightarrow t^{(m-3)}} \epsilon_{t^{(m-3)} \rightarrow t^{(m-4)}} \dots \epsilon_{t^{(1)} \rightarrow t}$. Notice that as $t(n-1) < 0$ and $t(n) < 0$, we have $x \in \mathcal{G}_{n-2}(\delta)$, which commutes with ψ_{n-1} .

Suppose $\rho(t^{(m)}, t^{(m-1)}) = (a, n)$ and $\rho(t^{(m-1)}, t^{(m-2)}) = (b, n-1)$ where $a \leq n-1$ and $b \leq n-2$. Hence we can write

$$\begin{aligned} \epsilon_{t^{(m)} \rightarrow t^{(m-1)}} &= e(\mathbf{i}_{t^{(m)}}) \epsilon_{2f} \epsilon_{2f+1} \dots \epsilon_a \psi_{a+1} \dots \psi_{n-1} e(\mathbf{i}_{t^{(m-1)}}), \\ \epsilon_{t^{(m-1)} \rightarrow t^{(m-2)}} &= e(\mathbf{i}_{t^{(m-1)}}) \epsilon_{2f-2} \epsilon_{2f-1} \dots \epsilon_b \psi_{b+1} \dots \psi_{n-2} e(\mathbf{i}_{t^{(m-2)}}). \end{aligned}$$

By (3.34), (3.38) and (3.39), we have

$$\begin{aligned}
\psi_{\mathfrak{t}(\lambda, f)\mathfrak{t}}\psi_{n-1} &= \psi_{\mathfrak{t}(\lambda, f)\mathfrak{s}}\epsilon_{\mathfrak{t}(m) \rightarrow \mathfrak{t}(m-1)}\epsilon_{\mathfrak{t}(m-1) \rightarrow \mathfrak{t}(m-2)}x\psi_{n-1} = \psi_{\mathfrak{t}(\lambda, f)\mathfrak{s}}\epsilon_{\mathfrak{t}(m) \rightarrow \mathfrak{t}(m-1)}\epsilon_{\mathfrak{t}(m-1) \rightarrow \mathfrak{t}(m-2)}\psi_{n-1}x \\
&= \psi_{\mathfrak{t}(\lambda, f)\mathfrak{s}}\left(\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}\right) \cdot \left(\epsilon_{2f-2}\epsilon_{2f-1} \dots \epsilon_b\psi_{b+1} \dots \psi_{n-1}\right)x \\
&= \psi_{\mathfrak{t}(\lambda, f)\mathfrak{s}}\left(\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_a\psi_{a+1} \dots \psi_{n-1}\right) \cdot \left(\epsilon_{2f-2}\epsilon_{2f-1} \dots \epsilon_{n-1}\right) \cdot (\psi_{n-2}\psi_{n-3} \dots \psi_b)x \\
&= \psi_{\mathfrak{t}(\lambda, f)\mathfrak{s}}\epsilon_{2f}\epsilon_{2f+1} \dots \epsilon_{n-1} \cdot \left(\epsilon_{2f-2}\epsilon_{2f-1} \dots \epsilon_{a-2}\psi_{a-1} \dots \psi_{n-3}\right) \cdot (\psi_{n-2}\psi_{n-3} \dots \psi_b)x.
\end{aligned}$$

As $\mathfrak{s} = h(\mathfrak{t}) \in \mathcal{T}_n^{ud}(\lambda)$ with head f and $\epsilon_{2f-2}\epsilon_{2f-1} \dots \epsilon_{a-2}\psi_{a-1} \dots \psi_{n-3}\psi_{n-2}\psi_{n-3} \dots \psi_b x \in \mathcal{G}_{n-1}(\delta)$, by Lemma 5.19 and Lemma 5.13, the Lemma holds. \square

Combining Lemma 5.23, Lemma 5.24 and Lemma 5.25, the next Lemma follows.

5.26. Lemma. Suppose $(\lambda, f) \in \mathcal{S}_n$ and $\mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$. Then the equality (5.1) holds when $a = \psi_{n-1}$.

Therefore, we have proved that (5.1) holds. Combining all the results of this section, we are ready to give the first main result of this paper.

5.27. Proposition. We have $\bigcup_{n \geq 1} \widehat{B}_n = \widehat{\mathcal{B}}$.

Proof. By the definition we have $\widehat{\mathcal{B}} \subseteq \bigcup_{n \geq 1} \widehat{B}_n$. It suffices to show that $\bigcup_{n \geq 1} \widehat{B}_n \subseteq \widehat{\mathcal{B}}$.

We have defined a total ordering $<$ on $\bigcup_{n \geq 1} \widehat{B}_n$. Hence we can list all the elements of $\bigcup_{n \geq 1} \widehat{B}_n$ in decreasing order as:

$$(\lambda_1, f_1) > (\lambda_2, f_2) > (\lambda_3, f_3) > \dots$$

We prove the Proposition by induction. For the base step it is easy to see that the maximal element in $\bigcup_{n \geq 1} \widehat{B}_n$ is $(\lambda_1, f_1) = ((1), 0) \in \widehat{B}_1$. Because $\mathcal{G}_1(\delta) \cong R$, it is obvious that for any $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}_1^{ud}(\lambda_1)$ and $a \in \mathcal{G}_1(\delta)$ we have

$$\psi_{\mathfrak{st}}a \equiv \sum_{\mathfrak{v} \in \mathcal{T}_1^{ud}(\lambda_1)} c_{\mathfrak{v}}\psi_{\mathfrak{sv}} \pmod{R_1^{>f_1}(\delta)},$$

which implies $(\lambda_1, f_1) \in \widehat{\mathcal{B}}$.

For induction step, assume $k > 1$ and $(\lambda_i, f_i) \in \widehat{\mathcal{B}}$ for any $1 \leq i \leq k-1$. Let $(\lambda, f) = (\lambda_k, f_k) \in \widehat{B}_n$. By the definition we have $(\lambda, f) \in \mathcal{S}_n$. For any $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$ and $a \in \mathcal{G}_n(\delta)$, by Lemma 5.13, Lemma 5.15, Lemma 5.22 and Lemma 5.26, we have

$$\psi_{\mathfrak{t}(\lambda, f)\mathfrak{t}}a \equiv \sum_{\mathfrak{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathfrak{v}}\psi_{\mathfrak{t}(\lambda, f)\mathfrak{v}} \pmod{R_n^{>f}(\delta)}.$$

Multiply $\epsilon_{\mathfrak{s}}^*\psi_{\mathfrak{s}}^*$ from left to the above equation. By Lemma 4.33 we have

$$\psi_{\mathfrak{st}}a \equiv \sum_{\mathfrak{v} \in \mathcal{T}_n^{ud}(\lambda)} c_{\mathfrak{v}}\psi_{\mathfrak{sv}} \pmod{R_n^{>f}(\delta)},$$

which implies $(\lambda, f) = (\lambda_k, f_k) \in \widehat{\mathcal{B}}$. Therefore we completes the induction process. Hence, for any $(\lambda, f) \in \bigcup_{n \geq 1} \widehat{B}_n$, we have $(\lambda, f) \in \widehat{\mathcal{B}}$, which proves $\bigcup_{n \geq 1} \widehat{B}_n \subseteq \widehat{\mathcal{B}}$. \square

5.28. Theorem. The algebra $\mathcal{G}_n(\delta)$ is spanned by $\{\psi_{\mathfrak{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)\}$.

Proof. Proposition 5.27 implies that for any $(\lambda, f) \in \widehat{B}_n$ and $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}_n^{ud}(\lambda)$, we have $\psi_{\mathfrak{st}}\mathcal{G}_n(\delta) \subseteq R_n(\delta)$. Hence we have $R_n(\delta)\mathcal{G}_n(\delta) \subseteq R_n(\delta)$. By Proposition 5.27 we have $\mathcal{S}_n = \widehat{B}_n$, which implies $e(\mathfrak{i}) \in R_n(\delta)$ for any $\mathfrak{i} \in P^n$ by Proposition 4.39. As $1 = \sum_{\mathfrak{i} \in P^n} e(\mathfrak{i})$ by (3.8), we have $1 \in R_n(\delta)$, which yields $\mathcal{G}_n(\delta) \subseteq R_n\mathcal{G}_n(\delta) \subseteq R_n(\delta)$.

Because $R_n(\delta) \subseteq \mathcal{G}_n(\delta)$ by the definition, we have $R_n(\delta) = \mathcal{G}_n(\delta)$, which proves the Theorem. \square

Theorem 5.28 shows that $\mathcal{G}_n(\delta)$ is a finite-dimensional R -space with $\dim \mathcal{G}_n(\delta) \leq (2n-1)!!$. The following results are directly implied.

Recall I^n is the subset of P^n containing all the residue sequence of up-down tableaux.

5.29. Corollary. Suppose $\mathfrak{i} \in P^n$. We have $e(\mathfrak{i}) = 0$ if $\mathfrak{i} \notin I^n$.

Proof. If $\mathfrak{i} \notin I^n$, we have $\mathcal{T}_n^{ud}(\mathfrak{i}) = \emptyset$. By Lemma 4.10, we have $\psi_{\mathfrak{ste}}(\mathfrak{i}) = 0$ for any up-down tableaux \mathfrak{s} and \mathfrak{t} . Hence by Theorem 5.28, we have $e(\mathfrak{i}) = e(\mathfrak{i})^2 = \sum_{\mathfrak{s}, \mathfrak{t}} c_{\mathfrak{st}}\psi_{\mathfrak{ste}}(\mathfrak{i}) = 0$. \square

5.30. Corollary. The elements $y_k \in \mathcal{G}_n(\delta)$ are nilpotent for all $1 \leq k \leq n$.

Proof. By Theorem 5.28, we have $\mathcal{G}_n(\delta) = R_n(\delta)$. Because there are finite elements in $\{\psi_{st} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$, there exists m such that $\deg \psi_{st} \leq m$. Hence for any homogeneous element $a \in \mathcal{G}_n(\delta)$, we have $\deg a \leq m$. Choose $N = \lfloor \frac{m}{2} \rfloor + 1$. For any $1 \leq k \leq n$ we have $\deg y_k^N > m$, which forces $y_k^N = 0$. \square

5.31. Remark. Theorem 5.28 shows that $\{\psi_{st} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$ spans $\mathcal{G}_n(\delta)$, and Proposition 5.27 shows that ψ_{st} 's have cellular-like property. Therefore, $\{\psi_{st} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)\}$ is a potential cellular basis of $\mathcal{G}_n(\delta)$. We will prove this result in Section 7.6.

6. A generating set of $\mathcal{B}_n(\delta)$

In the rest of this paper, we are going to prove that $\mathcal{G}_n(\delta) \cong \mathcal{B}_n(\delta)$. The first step of the proof is to define

$$G_n(\delta) = \{e(\mathbf{i}) \mid \mathbf{i} \in P^n\} \cup \{y_k \mid 1 \leq k \leq n\} \cup \{\psi_k \mid 1 \leq k \leq n-1\} \cup \{\epsilon_k \mid 1 \leq k \leq n-1\}$$

in $\mathcal{B}_n(\delta)$ and show that $G_n(\delta)$ generates $\mathcal{B}_n(\delta)$. Here we abuse the symbols and use $e(\mathbf{i})$, y_k , ψ_r and ϵ_r as elements in both $\mathcal{G}_n(\delta)$ and $\mathcal{B}_n(\delta)$. Then we construct a mapping from $\mathcal{G}_n(\delta)$ to $\mathcal{B}_n(\delta)$ by sending generators to generators and show this mapping is a surjective homomorphism. In this section, we construct the elements $e(\mathbf{i})$, y_k , ψ_r and ϵ_r in $\mathcal{B}_n(\delta)$ and show these elements generate $\mathcal{B}_n(\delta)$.

First we recall the definitions and notations we need for the rest of the paper, which have been introduced in Section 2.5. Recall R is a field with characteristic 0 and fix $\delta \in R$. Define $\mathbb{F} = R(x)$ to be the rational field with indeterminate x and $\mathcal{O} = R[x]_{(x-\delta)} = R[[x-\delta]]$. Let $\mathfrak{m} = (x-\delta)\mathcal{O} \subset \mathcal{O}$. Then \mathfrak{m} is a maximal ideal of \mathcal{O} and $R \cong \mathcal{O}/\mathfrak{m}$.

Let $\mathcal{B}_n^{\mathbb{F}}(x)$ and $\mathcal{B}_n^{\mathcal{O}}(x)$ be the Brauer algebras over \mathbb{F} and \mathcal{O} , respectively. Then $\mathcal{B}_n^{\mathbb{F}}(x) = \mathcal{B}_n^{\mathcal{O}}(x) \otimes_{\mathcal{O}} \mathbb{F}$ and $\mathcal{B}_n(\delta) \cong \mathcal{B}_n^{\mathcal{O}}(x) \otimes_{\mathcal{O}} R \cong \mathcal{B}_n^{\mathcal{O}}(x)/(x-\delta)\mathcal{B}_n^{\mathcal{O}}(x)$. In order to avoid confusion we will write the generators of $\mathcal{B}_n^{\mathcal{O}}(x)$ and $\mathcal{B}_n^{\mathbb{F}}(x)$ as $s_k^{\mathcal{O}}$ and $e_k^{\mathcal{O}}$ and generators of $\mathcal{B}_n(\delta)$ as s_k and e_k . Hence for any element $w \in \mathcal{B}_n(\delta)$, we write $w^{\mathcal{O}} = w \otimes_R 1_{\mathcal{O}} \in \mathcal{B}_n^{\mathcal{O}}(x)$, so that $w = w^{\mathcal{O}} \otimes_{\mathcal{O}} 1_R$.

Because $\mathcal{B}_n(\delta) \cong \mathcal{B}_n^{\mathcal{O}}(x) \otimes_{\mathcal{O}} R \cong \mathcal{B}_n^{\mathcal{O}}(x)/(x-\delta)\mathcal{B}_n^{\mathcal{O}}(x)$, if $x, y \in \mathcal{B}_n^{\mathcal{O}}(x)$ and we have $x \equiv y \pmod{(x-\delta)\mathcal{B}_n^{\mathcal{O}}(x)}$, then $x \otimes_{\mathcal{O}} 1_R = y \otimes_{\mathcal{O}} 1_R$ as elements of $\mathcal{B}_n(\delta)$. This observation will give us a way to extend the results of $\mathcal{B}_n^{\mathcal{O}}(x)$ to $\mathcal{B}_n(\delta)$.

6.1. Gelfand-Zetlin algebra of $\mathcal{B}_n(\delta)$

Following Okounkov-Vershik [15], define *Gelfand-Zetlin subalgebra* of $\mathcal{B}_n(\delta)$ to be the algebra \mathcal{L}_n generated by L_1, L_2, \dots, L_n . By the definition one can see that \mathcal{L}_n is a commutative subalgebra of $\mathcal{B}_n(\delta)$. Similarly we define $\mathcal{L}_n(\mathcal{O})$ in $\mathcal{B}_n^{\mathcal{O}}(x)$. In this subsection we define the idempotents $e(\mathbf{i})$ with $\mathbf{i} \in I^n$ and nilpotency elements y_k with $1 \leq k \leq n$ in $\mathcal{B}_n(\delta)$.

Let M be a finite dimensional $\mathcal{B}_n(\delta)$ -module. Similarly as in Brundan and Kleshchev [3, Section 3.1], the eigenvalues of each L_k on M belongs to P . So M decomposes as the direct sum $M = \bigoplus_{\mathbf{i} \in P^n} M_{\mathbf{i}}$ of weight spaces

$$M_{\mathbf{i}} = \{v \in M \mid (L_k - i_k)^N v = 0 \text{ for all } k = 1, 2, \dots, n \text{ and } N \gg 0\}.$$

We deduce that there is a system $\{e(\mathbf{i}) \mid \mathbf{i} \in P^n\}$ of mutually orthogonal idempotents in $\mathcal{B}_n(\delta)$ such that $Me(\mathbf{i}) = M_{\mathbf{i}}$ for each finite dimensional module M . In fact, $e(\mathbf{i})$ lies in \mathcal{L}_n .

Hu-Mathas [8, Proposition 4.8] proved the following result in the cyclotomic Hecke algebras. Their result can be directly extended to $\mathcal{B}_n(\delta)$ following the same proof.

6.1. Lemma (Hu-Mathas [8, Proposition 4.8]). *Suppose that $e(\mathbf{i}) \neq 0$ for some $\mathbf{i} \in P^n$ and let*

$$e(\mathbf{i})^{\mathcal{O}} = \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} \in \mathcal{B}_n^{\mathbb{F}}(x).$$

Then $e(\mathbf{i})^{\mathcal{O}} \in \mathcal{B}_n^{\mathcal{O}}(x)$ and $e(\mathbf{i}) = e(\mathbf{i})^{\mathcal{O}} \otimes_{\mathcal{O}} 1_R$.

By Lemma 6.1, it is straightforward that $e(\mathbf{i}) \neq 0$ only if $\mathbf{i} \in I^n$, i.e. \mathbf{i} is the residue sequence of an up-down tableau. Therefore, by defining $e(\mathbf{i}) = 0$ for $\mathbf{i} \notin I^n$, we construct a set of orthogonal elements $\{e(\mathbf{i})^{\mathcal{O}} \mid \mathbf{i} \in P^n\}$ in $\mathcal{B}_n^{\mathcal{O}}(x)$ and $\{e(\mathbf{i}) \mid \mathbf{i} \in P^n\}$ in $\mathcal{B}_n(\delta)$, such that $\sum_{\mathbf{i} \in P^n} e(\mathbf{i})^{\mathcal{O}} = \sum_{\mathbf{i} \in I^n} e(\mathbf{i})^{\mathcal{O}} = 1_{\mathcal{O}}$ and $\sum_{\mathbf{i} \in P^n} e(\mathbf{i}) = \sum_{\mathbf{i} \in I^n} e(\mathbf{i}) = 1_R$ by Proposition 2.19 and Lemma 6.1.

For an integer k with $1 \leq k \leq n$, define $y_k^{\mathcal{O}} := \sum_{\mathbf{i} \in I^n} (L_k^{\mathcal{O}} - i_k) e(\mathbf{i})^{\mathcal{O}} \in \mathcal{L}_n(\mathcal{O})$ and

$$y_k := y_k^{\mathcal{O}} \otimes_{\mathcal{O}} 1_R = \sum_{\mathbf{i} \in I^n} (L_k - i_k) e(\mathbf{i}) \in \mathcal{L}_n. \quad (6.1)$$

By Theorem 2.12, one can see that y_k is nilpotent in $\mathcal{B}_n(\delta)$. Hence for any polynomial $\phi(L_1, L_2, \dots, L_n)e(\mathbf{i}) \in \mathcal{L}_n$ with $\phi(i_1, i_2, \dots, i_n) \neq 0$, we can define a element $\phi(L_1, \dots, L_n)^{-1} \in \mathcal{L}_n$ such that

$$\phi(L_1, L_2, \dots, L_n)\phi(L_1, L_2, \dots, L_n)^{-1}e(\mathbf{i}) = \phi(L_1, L_2, \dots, L_n)^{-1}\phi(L_1, L_2, \dots, L_n)e(\mathbf{i}) = e(\mathbf{i}).$$

Suppose $\phi(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ is a polynomial. Define $R^\times = \{r \in R \mid r \neq 0\}$. The next result is the extended version of Hu-Mathas [9, Proposition 4.6], followed by the same proof.

6.2. Lemma. Suppose $\mathbf{i} \in I^n$ and $\phi(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ is a polynomial. If $\phi(i_1, \dots, i_n) \in R^\times$, we have

$$\sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{\phi(c_t(1), \dots, c_t(n))} \frac{f_t}{\gamma_t} \in \mathcal{L}_n(\mathcal{O}).$$

6.3. Lemma. Suppose $\mathbf{i} \in I^n$ and $\phi(x_1, \dots, x_n) \in R(x_1, \dots, x_n)$ is a rational function. If $\phi(i_1, \dots, i_n) \in R$, we have

$$\sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \phi(c_t(1), \dots, c_t(n)) \frac{f_t}{\gamma_t} \in \mathcal{L}_n(\mathcal{O}).$$

Proof. Because ϕ is a rational function, there are two polynomials ϕ_1 and ϕ_2 such that $\phi = \phi_1/\phi_2$. It is obvious that $\phi(i_1, \dots, i_n) \in R$ if and only if $\phi_2(i_1, \dots, i_n) \in R^\times$. Hence by Lemma 6.2, we have

$$\sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{\phi_2(c_t(1), \dots, c_t(n))} \frac{f_t}{\gamma_t} \in \mathcal{L}_n(\mathcal{O}).$$

Hence, because ϕ_1 is a polynomial, we have $\phi_1(L_1^\mathcal{O}, \dots, L_n^\mathcal{O}) \in \mathcal{L}_n(\mathcal{O})$. So

$$\phi_1(L_1^\mathcal{O}, \dots, L_n^\mathcal{O}) \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{\phi_2(c_t(1), \dots, c_t(n))} \frac{f_t}{\gamma_t} = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \phi(c_t(1), \dots, c_t(n)) \frac{f_t}{\gamma_t} \in \mathcal{L}_n(\mathcal{O}),$$

which completes the proof. \square

Let ϕ be a polynomial in $R[x_1, \dots, x_n]$ satisfying the assumptions of Lemma 6.2. Then

$$\phi(L_1^\mathcal{O}, \dots, L_n^\mathcal{O}) \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{\phi(c_t(1), \dots, c_t(n))} \frac{f_t}{\gamma_t} = e(\mathbf{i})^\mathcal{O} = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{\phi(c_t(1), \dots, c_t(n))} \frac{f_t}{\gamma_t} \phi(L_1^\mathcal{O}, \dots, L_n^\mathcal{O}).$$

Abusing notations, in this situation we write

$$\frac{1}{\phi(L_1^\mathcal{O}, \dots, L_n^\mathcal{O})} e(\mathbf{i})^\mathcal{O} = e(\mathbf{i})^\mathcal{O} \frac{1}{\phi(L_1^\mathcal{O}, \dots, L_n^\mathcal{O})} = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{\phi(c_t(1), \dots, c_t(n))} \frac{f_t}{\gamma_t} \in \mathcal{L}_n(\mathcal{O}).$$

Similarly, let ϕ be a rational function in $R(x_1, \dots, x_n)$ satisfying the assumptions of Lemma 6.3. Then we write

$$\phi(L_1^\mathcal{O}, \dots, L_n^\mathcal{O}) e(\mathbf{i})^\mathcal{O} = e(\mathbf{i})^\mathcal{O} \phi(L_1^\mathcal{O}, \dots, L_n^\mathcal{O}) = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \phi(c_t(1), \dots, c_t(n)) \frac{f_t}{\gamma_t}.$$

6.2. Modification terms $P_k(\mathbf{i})$, $Q_k(\mathbf{i})$ and $V_k(\mathbf{i})$

In this subsection we define three elements $P_k(\mathbf{i})$, $Q_k(\mathbf{i})$ and $V_k(\mathbf{i})$ in $\mathcal{B}_n(\delta)$, which are essential when we define ψ_k and ϵ_k in $\mathcal{B}_n(\delta)$. These terms are defined so that the actions of ψ_k and ϵ_k 's on seminormal forms of $\mathcal{B}_n(\delta)$ are well-behaved (cf. Lemma 7.2 and Lemma 7.4).

Let x_1, \dots, x_n be invariants. For each $\mathbf{i} \in I^n$ and rational function ϕ , we say $\phi(x_1, \dots, x_{k-1}, \frac{x_k - x_{k+1}}{2}) \in R(x_1, \dots, x_{k+1})$ is *invertible over \mathbf{i}* if $\phi(i_1, i_2, \dots, i_{k-1}, (i_k - i_{k+1})/2) \in R^\times$. It is obvious by the definition that $\phi_1, \phi_2 \in R(x_1, \dots, x_{k+1})$ invertible over \mathbf{i} implies $\phi_1 \cdot \phi_2 \in R(x_1, \dots, x_{k+1})$ invertible over \mathbf{i} .

First we define the elements $P_k(\mathbf{i})$ and $Q_k(\mathbf{i})$. For $1 \leq k \leq n-1$ and $1 \leq r \leq k-1$, define

$$\begin{aligned} L_{k,r} &= \left\{ \frac{x_k - x_{k+1}}{2} + x_r + 1, \frac{1}{\frac{x_k - x_{k+1}}{2} - x_r + 1}, -(\frac{x_k - x_{k+1}}{2} - x_r), -(\frac{1}{\frac{x_k - x_{k+1}}{2} + x_r}) \right\} \subset R(x_1, \dots, x_{k+1}), \\ R_{k,r} &= \left\{ -(\frac{x_k - x_{k+1}}{2} + x_r - 1), -\frac{1}{\frac{x_k - x_{k+1}}{2} - x_r - 1}, \frac{x_k - x_{k+1}}{2} - x_r, \frac{1}{\frac{x_k - x_{k+1}}{2} + x_r} \right\} \subset R(x_1, \dots, x_{k+1}), \\ S_k &= \left\{ -\frac{1}{\frac{x_k - x_{k+1}}{2} - x_1}, -\frac{1}{x_k - x_{k+1}}, x_k - x_{k+1} + 1 \right\} \subset R(x_1, x_k, x_{k+1}), \\ T_k &= \left\{ \frac{x_k - x_{k+1}}{2} + x_1 \right\} \subset R(x_1, x_k, x_{k+1}). \end{aligned}$$

For any $\mathbf{i} \in P^n$, we define

$$\begin{aligned} L_{k,r}(\mathbf{i}) &= \{w \in L_{k,r} \mid w \text{ is invertible over } \mathbf{i}\}, & S_k(\mathbf{i}) &= \{w \in S_k \mid w \text{ is invertible over } \mathbf{i}\}, \\ R_{k,r}(\mathbf{i}) &= \{w \in R_{k,r} \mid w \text{ is invertible over } \mathbf{i}\}, & T_k(\mathbf{i}) &= \{w \in T_k \mid w \text{ is invertible over } \mathbf{i}\}. \end{aligned}$$

Let

$$\begin{aligned} P_k^{\mathbf{i}}(x_1, \dots, x_{k-1}, \frac{x_k - x_{k+1}}{2}) &:= \prod_{w \in S_k(\mathbf{i})} w \prod_{r=1}^{k-1} \left(\prod_{w \in L_{k,r}(\mathbf{i})} w \right) \in R(x_1, \dots, x_{k+1}), \\ Q_k^{\mathbf{i}}(x_1, \dots, x_{k-1}, \frac{x_k - x_{k+1}}{2}) &:= \prod_{w \in T_k(\mathbf{i})} w \prod_{r=1}^{k-1} \left(\prod_{w \in R_{k,r}(\mathbf{i})} w \right) \in R(x_1, \dots, x_{k+1}). \end{aligned}$$

By the definitions, one can see that $P_k^{\mathbf{i}}, Q_k^{\mathbf{i}} \in R(x_1, \dots, x_{k+1})$ are invertible over \mathbf{i} . Hence

$$\begin{aligned} P_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, \frac{i_k - i_{k+1}}{2}) &\in R^\times, & Q_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, \frac{i_k - i_{k+1}}{2}) &\in R^\times, \\ P_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, \frac{i_k - i_{k+1}}{2})^{-1} &\in R^\times, & Q_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, \frac{i_k - i_{k+1}}{2})^{-1} &\in R^\times. \end{aligned} \quad (6.2)$$

For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we define $P_k(\mathbf{t}), Q_k(\mathbf{t}) \in \mathbb{F}$ by

$$\begin{aligned} P_k(\mathbf{t}) &= P_k^{\mathbf{i}}(c_{\mathbf{t}}(1), \dots, c_{\mathbf{t}}(k-1), (c_{\mathbf{t}}(k) - c_{\mathbf{t}}(k+1))/2), \\ Q_k(\mathbf{t}) &= Q_k^{\mathbf{i}}(c_{\mathbf{t}}(1), \dots, c_{\mathbf{t}}(k-1), (c_{\mathbf{t}}(k) - c_{\mathbf{t}}(k+1))/2). \end{aligned}$$

Define $P_k^{\mathcal{O}}(\mathbf{i}), Q_k^{\mathcal{O}}(\mathbf{i}), P_k^{\mathcal{O}}(\mathbf{i})^{-1}, Q_k^{\mathcal{O}}(\mathbf{i})^{-1} \in \mathcal{B}_n^{\mathbb{F}}(x)$ by

$$\begin{aligned} P_k^{\mathcal{O}}(\mathbf{i}) &= \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} P_k(\mathbf{t}) \frac{f_{\mathbf{i}}}{\gamma_{\mathbf{i}}}, & Q_k^{\mathcal{O}}(\mathbf{i}) &= \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} Q_k(\mathbf{t}) \frac{f_{\mathbf{i}}}{\gamma_{\mathbf{i}}}, \\ P_k^{\mathcal{O}}(\mathbf{i})^{-1} &= \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{P_k(\mathbf{t})} \frac{f_{\mathbf{i}}}{\gamma_{\mathbf{i}}}, & Q_k^{\mathcal{O}}(\mathbf{i})^{-1} &= \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{Q_k(\mathbf{t})} \frac{f_{\mathbf{i}}}{\gamma_{\mathbf{i}}}, \end{aligned} \quad (6.3)$$

and by (6.2) and Lemma 6.3, we have $P_k^{\mathcal{O}}(\mathbf{i}), Q_k^{\mathcal{O}}(\mathbf{i}), P_k^{\mathcal{O}}(\mathbf{i})^{-1}, Q_k^{\mathcal{O}}(\mathbf{i})^{-1} \in \mathcal{L}_n(\mathcal{O})$.

6.4. Definition. Suppose $\mathbf{i} \in I^n$ and $1 \leq k \leq n-1$. We define elements $P_k(\mathbf{i}), Q_k(\mathbf{i}), P_k(\mathbf{i})^{-1}, Q_k(\mathbf{i})^{-1} \in \mathcal{L}_n$ by

$$\begin{aligned} P_k(\mathbf{i}) &= P_k^{\mathcal{O}}(\mathbf{i}) \otimes_{\mathcal{O}} 1_R, & Q_k(\mathbf{i}) &= Q_k^{\mathcal{O}}(\mathbf{i}) \otimes_{\mathcal{O}} 1_R, \\ P_k(\mathbf{i})^{-1} &= P_k^{\mathcal{O}}(\mathbf{i})^{-1} \otimes_{\mathcal{O}} 1_R, & Q_k(\mathbf{i})^{-1} &= Q_k^{\mathcal{O}}(\mathbf{i})^{-1} \otimes_{\mathcal{O}} 1_R. \end{aligned}$$

The next Lemma is directly implied by (6.3).

6.5. Lemma. Suppose $\mathbf{i} \in I^n$ and $1 \leq k \leq n-1$. We have

$$\begin{aligned} P_k(\mathbf{i})P_k(\mathbf{i})^{-1} &= P_k(\mathbf{i})^{-1}P_k(\mathbf{i}) = e(\mathbf{i}), \\ Q_k(\mathbf{i})Q_k(\mathbf{i})^{-1} &= Q_k(\mathbf{i})^{-1}Q_k(\mathbf{i}) = e(\mathbf{i}). \end{aligned}$$

Next, for each $1 \leq k \leq n-1$ and $\mathbf{i} \in I^n$ with $i_k = i_{k+1}$, we define element $V_k(\mathbf{i})$. First we define a rational function

$$V_k^{\mathbf{i}}(x_1, \dots, x_{k-1}, \frac{x_k - x_{k+1}}{2}) := \frac{P_k^{\mathbf{i}}Q_k^{\mathbf{i}} - 1}{x_k - x_{k+1}} \in R(x_1, \dots, x_{k+1}).$$

The next Lemma shows that $V_k^{\mathbf{i}}$ is invertible over \mathbf{i} .

6.6. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ with $i_k = i_{k+1}$. Then we have $V_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, \frac{i_k - i_{k+1}}{2}) \in R$.

Proof. As $i_k = i_{k+1}$, we have $\frac{i_k - i_{k+1}}{2} = 0$. Because all the factors of $P_k^{\mathbf{i}}$ and $Q_k^{\mathbf{i}}$ are invertible over \mathbf{i} , we can write

$$P_k^{\mathbf{i}}Q_k^{\mathbf{i}} = \sum_{i=0}^{\infty} c_i(x_1, \dots, x_{k-1}) \left(\frac{x_k - x_{k+1}}{2} \right)^i,$$

where $c_i(x_1, \dots, x_{k-1}) \in R(x_1, \dots, x_{k-1})$ and $c_i(i_1, \dots, i_{k-1}) \in R$. Hence

$$V_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, \frac{x_k - x_{k+1}}{2}) = \frac{c_1(i_1, \dots, i_{k-1}) - 1}{x_k - x_{k+1}} + \sum_{i=0}^{\infty} \frac{c_{i+1}(i_1, \dots, i_{k-1})}{2} \left(\frac{x_k - x_{k+1}}{2} \right)^i,$$

and $V_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, \frac{i_k - i_{k+1}}{2}) \in R$ if and only if $c_1(i_1, \dots, i_{k-1}) = 1$. By the definitions of $P_k^{\mathbf{i}}$ and $Q_k^{\mathbf{i}}$, we have

$$c_1(i_1, \dots, i_{k-1}) = P_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, 0)Q_k^{\mathbf{i}}(i_1, \dots, i_{k-1}, 0) = 1,$$

which completes the proof. \square

Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. Define $V_k(\mathbf{t}) = V_k^{\mathbf{i}}(c_{\mathbf{t}}(1), \dots, c_{\mathbf{t}}(k-1), \frac{c_{\mathbf{t}}(k)-c_{\mathbf{t}}(k+1)}{2})$. We define $V_k^{\mathcal{O}}(\mathbf{i}) \in \mathcal{B}_n^{\mathcal{O}}(x)$ by

$$V_k^{\mathcal{O}}(\mathbf{i}) = \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} V_k(\mathbf{t}) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}},$$

and by Lemma 6.3 and Lemma 6.6, we have $V_k^{\mathcal{O}}(\mathbf{i}) \in \mathcal{L}_n(\mathcal{O})$.

6.7. Definition. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ with $i_k = i_{k+1}$. We define the element $V_k(\mathbf{i}) \in \mathcal{L}_n$ by $V_k(\mathbf{i}) := V_k^{\mathcal{O}}(\mathbf{i}) \otimes_{\mathcal{O}} 1_R$.

The next Lemma shows the connections of $V_k(\mathbf{t})$ with $P_k(\mathbf{t})$ and $Q_k(\mathbf{t})$.

6.8. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ with $i_k = i_{k+1}$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $P_k(\mathbf{t})Q_k(\mathbf{t}) = (c_{\mathbf{t}}(k) - c_{\mathbf{t}}(k+1))V_k(\mathbf{t}) + 1$.

Proof. By the definition of $V_k^{\mathbf{i}}$, we have $P_k^{\mathbf{i}}Q_k^{\mathbf{i}} = (x_k - x_{k+1})V_k^{\mathbf{i}} + 1$. Hence the Lemma follows straightforward. \square

In the rest of this subsection we introduce some of the properties of $P_k(\mathbf{i})$, $Q_k(\mathbf{i})$ and $V_k(\mathbf{i})$, or more precisely, properties of $P_k(\mathbf{t})$, $Q_k(\mathbf{t})$ and $V_k(\mathbf{t})$ for up-down tableaux \mathbf{t} . These results will be used frequently in the rest of this paper when we derive the relations of the generators of $\mathcal{B}_n(\delta)$. These properties make the actions of the ψ_k 's and ϵ_k 's on the seminormal basis well-behaved and they are the reasons why we define the modification terms in such a way.

For any rational function $w \in R(x_1, \dots, x_n)$, we denote $w(\mathbf{t}) = w(c_{\mathbf{t}}(1), \dots, c_{\mathbf{t}}(n))$.

6.9. Lemma. Suppose $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. For $1 \leq k \leq n-1$, if $\mathbf{u} = \mathbf{t} \cdot s_k$ exists, then we have

$$P_k(\mathbf{t})^{-1}Q_k(\mathbf{u})^{-1} = \begin{cases} \frac{1}{1-c_{\mathbf{u}}(k)+c_{\mathbf{u}}(k+1)}, & \text{if } i_{k+1} = i_k, \\ c_{\mathbf{u}}(k) - c_{\mathbf{u}}(k+1), & \text{if } i_{k+1} = i_k - 1, \\ \frac{c_{\mathbf{u}}(k)-c_{\mathbf{u}}(k+1)}{1-c_{\mathbf{u}}(k)+c_{\mathbf{u}}(k+1)}, & \text{if } i_{k+1} \neq i_k, i_k - 1. \end{cases}$$

Proof. Because $\mathbf{t} = \mathbf{u} \cdot s_k$, we have $c_{\mathbf{u}}(r) = c_{\mathbf{t}}(r)$ for $1 \leq r \leq k-1$ and $c_{\mathbf{u}}(k) - c_{\mathbf{u}}(k+1) = -(c_{\mathbf{t}}(k) - c_{\mathbf{t}}(k+1))$. Hence for any $1 \leq \ell \leq k-1$, we have

$$\left(\prod_{w \in L_{k,r}(\mathbf{i} \cdot s_k)} w(\mathbf{t}) \right) \left(\prod_{w \in R_{k,r}(\mathbf{i})} w(\mathbf{u}) \right) = 1.$$

Therefore, by the definition of $P_k(\mathbf{t})$ and $Q_k(\mathbf{u})$, we have

$$P_k(\mathbf{t})Q_k(\mathbf{u}) = \prod_{w \in S_k(\mathbf{j})} w(\mathbf{t}) \prod_{w \in T_k(\mathbf{i})} w(\mathbf{u}) = \begin{cases} 1 - c_{\mathbf{u}}(k) + c_{\mathbf{u}}(k+1), & \text{if } i_{k+1} = i_k, \\ \frac{1}{c_{\mathbf{u}}(k) - c_{\mathbf{u}}(k+1)}, & \text{if } i_{k+1} = i_k - 1, \\ \frac{1 - c_{\mathbf{u}}(k) + c_{\mathbf{u}}(k+1)}{c_{\mathbf{u}}(k) - c_{\mathbf{u}}(k+1)}, & \text{otherwise.} \end{cases}$$

Hence the Lemma follows. \square

6.10. Lemma. Suppose $\mathbf{i} \in I^n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. If $\mathbf{t}(k-1) = \mathbf{t}(k+1) = -\mathbf{t}(k)$, we have

$$Q_k(\mathbf{t})P_{k-1}(\mathbf{t}) = P_k(\mathbf{t})Q_{k-1}(\mathbf{t}) = 1,$$

for $2 \leq k \leq n-1$.

Proof. By the construction of \mathbf{t} , we have $c_{\mathbf{t}}(k-1) = c_{\mathbf{t}}(k+1) = -c_{\mathbf{t}}(k)$. Hence we have $c_{\mathbf{t}}(k) = (c_{\mathbf{t}}(k) - c_{\mathbf{t}}(k+1))/2$ and $c_{\mathbf{t}}(k-1) = (c_{\mathbf{t}}(k-1) - c_{\mathbf{t}}(k))/2$. Because $c_{\mathbf{t}}(k-1) = -c_{\mathbf{t}}(k)$, for $1 \leq r \leq k-2$, we have

$$\left(\prod_{w \in L_{k,r}(\mathbf{i})} w(\mathbf{t}) \right) \left(\prod_{w \in R_{k-1,r}(\mathbf{i})} w(\mathbf{t}) \right) = 1.$$

Hence, by the definition of $P_k(\mathbf{t})$ and $Q_{k-1}(\mathbf{t})$, we have

$$P_k(\mathbf{t})Q_{k-1}(\mathbf{t}) = \prod_{w \in S_k(\mathbf{i})} w(\mathbf{t}) \prod_{w \in T_{k-1}(\mathbf{i})} w(\mathbf{t}) \left(\prod_{w \in L_{k,k-1}(\mathbf{i})} w(\mathbf{t}) \right).$$

Then $P_k(\mathbf{t})Q_{k-1}(\mathbf{t})$ is the product of the non-invertible elements of

$$\left\{ -\frac{1}{c_{\mathbf{t}}(k) - c_{\mathbf{t}}(1)}, -\frac{1}{2c_{\mathbf{t}}(k)}, 2c_{\mathbf{t}}(k) + 1, 1, \frac{1}{2c_{\mathbf{t}}(k) + 1}, -2c_{\mathbf{t}}(k), -c_{\mathbf{t}}(k) + c_{\mathbf{t}}(1) \right\}.$$

By direct calculation, we have $P_k(\mathbf{t})Q_{k-1}(\mathbf{t}) = 1$. $Q_k(\mathbf{t})P_{k-1}(\mathbf{t}) = 1$ follows by the similar argument. \square

Recall that

$$a_k(\mathbf{i}) = \begin{cases} \# \{1 \leq r \leq k-1 \mid i_r \in \{-1, 1\}\} + 1 + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } (i_k - i_{k+1})/2 = 0, \\ \# \{1 \leq r \leq k-1 \mid i_r \in \{-1, 1\}\} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } (i_k - i_{k+1})/2 = 1, \\ \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{if } (i_k - i_{k+1})/2 = 1/2, \\ \# \{1 \leq r \leq k-1 \mid i_r \in \{\frac{i_k - i_{k+1}}{2}, \frac{i_k - i_{k+1}}{2} - 1, -\frac{i_k - i_{k+1}}{2}, -\frac{i_k - i_{k+1}}{2} + 1\}\} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}, & \text{otherwise.} \end{cases}$$

For $1 \leq k \leq n-1$ and $1 \leq r \leq k-1$, define

$$\begin{aligned} \widehat{L}_{k,r} &= \left\{ \frac{x_k - x_{k+1}}{2} + x_r + 1, \frac{1}{\frac{x_k - x_{k+1}}{2} - x_r + 1}, \frac{x_k - x_{k+1}}{2} - x_r, \frac{1}{\frac{x_k - x_{k+1}}{2} + x_r} \right\} \subset R(x_1, \dots, x_{k+1}), \\ \widehat{R}_{k,r} &= \left\{ \frac{x_k - x_{k+1}}{2} + x_r - 1, \frac{1}{\frac{x_k - x_{k+1}}{2} - x_r - 1}, \frac{x_k - x_{k+1}}{2} - x_r, \frac{1}{\frac{x_k - x_{k+1}}{2} + x_r} \right\} \subset R(x_1, \dots, x_{k+1}), \\ \widehat{S}_k &= \left\{ \frac{1}{\frac{x_k - x_{k+1}}{2} - x_1}, \frac{1}{x_k - x_{k+1}}, x_k - x_{k+1} + 1 \right\} \subset R(x_1, x_k, x_{k+1}), \\ \widehat{T}_k &= \left\{ \frac{x_k - x_{k+1}}{2} + x_1 \right\} \subset R(x_1, x_k, x_{k+1}). \end{aligned}$$

For any residue sequence $\mathbf{i} \in I^n$, define

$$\begin{aligned} \widehat{L}_{k,r}(\mathbf{i}) &= \{w \in \widehat{L}_{k,r} \mid w \text{ is invertible over } \mathbf{i}\}, & \widehat{S}_k(\mathbf{i}) &= \{w \in \widehat{S}_k \mid w \text{ is invertible over } \mathbf{i}\}, \\ \widehat{R}_{k,r}(\mathbf{i}) &= \{w \in \widehat{R}_{k,r} \mid w \text{ is invertible over } \mathbf{i}\}, & \widehat{T}_k(\mathbf{i}) &= \{w \in \widehat{T}_k \mid w \text{ is invertible over } \mathbf{i}\}. \end{aligned}$$

For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, let

$$\widehat{P}_k(\mathbf{t}) := \prod_{w \in \widehat{S}_k(\mathbf{i})} w(\mathbf{t}) \prod_{r=1}^{k-1} \left(\prod_{w \in \widehat{L}_{k,r}(\mathbf{i})} w(\mathbf{t}) \right), \quad \text{and} \quad \widehat{Q}_k(\mathbf{t}) := \prod_{w \in \widehat{T}_k(\mathbf{i})} w(\mathbf{t}) \prod_{r=1}^{k-1} \left(\prod_{w \in \widehat{R}_{k,r}(\mathbf{i})} w(\mathbf{t}) \right).$$

6.11. Lemma. Suppose $\mathbf{i} \in I^n$ and $1 \leq k \leq n-1$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $P_k(\mathbf{t})Q_k(\mathbf{t}) = (-1)^{a_k(\mathbf{i})} \widehat{P}_k(\mathbf{t}) \widehat{Q}_k(\mathbf{t})$.

Proof. For $1 \leq r \leq k-1$, define $b_{k,r}(\mathbf{i}) = \delta_{i_r, \frac{i_k - i_{k+1}}{2}} + \delta_{i_r, \frac{i_k - i_{k+1}}{2} - 1} + \delta_{i_r, -\frac{i_k - i_{k+1}}{2}} + \delta_{i_r, -\frac{i_k - i_{k+1}}{2} + 1}$. By comparing $a_k(\mathbf{i})$ and $\sum_{r=1}^{k-1} b_{k,r}(\mathbf{i})$, it is easy to see that

$$(-1)^{a_k(\mathbf{i})} = (-1)^{\sum_{r=1}^{k-1} b_{k,r}(\mathbf{i}) + \delta_{\frac{i_k - i_{k+1}}{2}, 0} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}}. \quad (6.4)$$

By the definitions of $L_{k,r}(\mathbf{i})$, $\widehat{L}_{k,r}(\mathbf{i})$, $R_{k,r}(\mathbf{i})$ and $\widehat{R}_{k,r}(\mathbf{i})$, for any $1 \leq r \leq k-1$, we have

$$\prod_{w \in L_{k,r}(\mathbf{i})} w(\mathbf{t}) \prod_{w \in R_{k,r}(\mathbf{i})} w(\mathbf{t}) = (-1)^{b_{k,r}(\mathbf{i})} \prod_{w \in \widehat{L}_{k,r}(\mathbf{i})} w(\mathbf{t}) \prod_{w \in \widehat{R}_{k,r}(\mathbf{i})} w(\mathbf{t}),$$

which implies

$$\prod_{r=1}^{k-1} \left(\prod_{w \in L_{k,r}(\mathbf{i})} w(\mathbf{t}) \prod_{w \in R_{k,r}(\mathbf{i})} w(\mathbf{t}) \right) = (-1)^{\sum_{r=1}^{k-1} b_{k,r}(\mathbf{i})} \prod_{r=1}^{k-1} \left(\prod_{w \in \widehat{L}_{k,r}(\mathbf{i})} w(\mathbf{t}) \prod_{w \in \widehat{R}_{k,r}(\mathbf{i})} w(\mathbf{t}) \right). \quad (6.5)$$

By the definitions of $S_k(\mathbf{i})$, $\widehat{S}_k(\mathbf{i})$, $T_k(\mathbf{i})$ and $\widehat{T}_k(\mathbf{i})$, we have

$$\prod_{w \in S_k(\mathbf{i})} w(\mathbf{t}) \prod_{w \in T_k(\mathbf{i})} w(\mathbf{t}) = (-1)^{\delta_{\frac{i_k - i_{k+1}}{2}, 0} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}} \prod_{w \in \widehat{S}_k(\mathbf{i})} w(\mathbf{t}) \prod_{w \in \widehat{T}_k(\mathbf{i})} w(\mathbf{t}). \quad (6.6)$$

Combining (6.5) and (6.6), we have

$$P_k(\mathbf{t})Q_k(\mathbf{t}) = (-1)^{\sum_{r=1}^{k-1} b_{k,r}(\mathbf{i}) + \delta_{\frac{i_k - i_{k+1}}{2}, 0} + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}} \widehat{P}_k(\mathbf{t}) \widehat{Q}_k(\mathbf{t}).$$

Hence the Lemma follows by (6.4). \square

6.12. Lemma. Suppose $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ with $i_k + i_{k+1} = 0$ for $1 \leq k \leq n-2$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$. Then we have

$$P_k(\mathbf{t})Q_k(\mathbf{t}) = \begin{cases} (-1)^{a_k(\mathbf{i})} \cdot e_k(\mathbf{t}, \mathbf{t}), & \text{if } \mathbf{i} \in I_{k,0}^n, \\ (-1)^{a_k(\mathbf{i})} \cdot 2(c_t(k) - i_k) e_k(\mathbf{t}, \mathbf{t}), & \text{if } \mathbf{i} \in I_{k,-}^n, \\ (-1)^{a_k(\mathbf{i})} \cdot \frac{1}{2(c_t(k) - i_k)} e_k(\mathbf{t}, \mathbf{t}), & \text{if } \mathbf{i} \in I_{k,+}^n. \end{cases}$$

Proof. By Lemma 3.3, we have

$$(2u+1) \sum_{\substack{s \leq t \\ s \neq t}} \frac{u+c_s(k)}{u-c_s(k)} = (2u+1) \frac{u-c_t(k)}{u+c_t(k)} \frac{u+c_t(1)}{u-c_t(1)} \prod_{r=1}^{k-1} \frac{(u+c_t(r))^2-1}{(u-c_t(r))^2-1} \frac{(u-c_t(r))^2}{(u+c_t(r))^2}. \quad (6.7)$$

Define $f(x_1, x_2, \dots, x_{k-1}, \frac{x_k-x_{k+1}}{2})$ to be a rational function in $R(x_1, \dots, x_{k+1})$ obtained by removing all factors which are non-invertible over \mathbf{i} from

$$(x_k - x_{k+1} + 1) \frac{1}{x_k - x_{k+1}} \frac{\frac{x_k-x_{k+1}}{2} + x_1}{\frac{x_k-x_{k+1}}{2} - x_1} \prod_{r=1}^{k-1} \frac{(\frac{x_k-x_{k+1}}{2} + x_r) + 1}{(\frac{x_k-x_{k+1}}{2} - x_r) + 1} \frac{(\frac{x_k-x_{k+1}}{2} + x_r) - 1}{(\frac{x_k-x_{k+1}}{2} - x_r) - 1} \frac{(\frac{x_k-x_{k+1}}{2} - x_r)^2}{(\frac{x_k-x_{k+1}}{2} + x_r)^2}. \quad (6.8)$$

Because $c_t(k) + c_t(k+1) = 0$, we have $\frac{c_t(k)-c_t(k+1)}{2} = c_t(k)$. If w is a factor of (6.8) which is non-invertible over \mathbf{i} , one can see that $w(\mathbf{t}) \in \{2(c_t(k) - i_k), \frac{1}{2(c_t(k) - i_k)}, 0\}$. Hence, by (6.7) we have

$$f(c_t(1), \dots, c_t(k-1), \frac{c_t(k)-c_t(k+1)}{2}) = f(c_t(1), \dots, c_t(k)) = (2(c_t(k) - i_k))^\ell e_k(\mathbf{t}, \mathbf{t}) \quad (6.9)$$

for some $\ell \in \mathbb{Z}$.

By the definition of f , we can see that $f(x_1, \dots, x_{k-1}, \frac{x_k-x_{k+1}}{2}) = \widehat{P}_k^{\mathbf{i}}(x_1, \dots, x_{k-1}, \frac{x_k-x_{k+1}}{2}) \widehat{Q}_k^{\mathbf{i}}(x_1, \dots, x_{k-1}, \frac{x_k-x_{k+1}}{2})$. Hence by Lemma 6.11 and (6.9), we have

$$P_k(\mathbf{t}) Q_k(\mathbf{t}) = (-1)^{a_k(\mathbf{i})} f(c_t(1), \dots, c_t(k)) = (-1)^{a_k(\mathbf{i})} (2(c_t(k) - i_k))^\ell e_k(\mathbf{t}, \mathbf{t}), \quad (6.10)$$

where $\ell \in \mathbb{Z}$.

By Lemma 2.17, we have

$$\sum_{\substack{s \leq t \\ s \neq t}} \frac{c_t(k) + c_s(k)}{c_t(k) - c_s(k)} = (2(c_t(k) - i_k))^{|A\mathcal{R}_A(-i_k)| - |A\mathcal{R}_A(-i_k)| + 1} v,$$

for some v invertible in \mathcal{O} . Hence, by the definitions of $I_{k,0}^n, I_{k,-}^n, I_{k,+}^n$ and (3.5) - (3.7), we have

$$e_k(\mathbf{t}, \mathbf{t}) = (2c_t(k) + 1) \sum_{\substack{s \leq t \\ s \neq t}} \frac{c_t(k) + c_s(k)}{c_t(k) - c_s(k)} = \begin{cases} v, & \text{if } \mathbf{i} \in I_{k,0}^n, \\ \frac{1}{2(c_t(k) - i_k)} v, & \text{if } \mathbf{i} \in I_{k,-}^n, \\ 2(c_t(k) - i_k) v, & \text{if } \mathbf{i} \in I_{k,+}^n, \end{cases}$$

for some v invertible in \mathcal{O} . Hence as $P_k(\mathbf{t}), Q_k(\mathbf{t})$ are invertible in \mathcal{O} , by (6.10) we complete the proof. \square

6.13. Lemma. Suppose $1 \leq r, k \leq n-1$ with $r < k-1$ and \mathbf{t} is an up-down tableau. If $\mathbf{s} = \mathbf{t} \cdot s_r$ exists, we have $P_k(\mathbf{t}) = P_k(\mathbf{s})$ and $Q_k(\mathbf{t}) = Q_k(\mathbf{s})$.

Proof. Because $\mathbf{s} = \mathbf{t} \cdot s_r$, we have $\mathbf{t}(\ell) = \mathbf{s}(\ell)$ for $1 \leq \ell \leq n$ and $\ell \neq r, r+1$, and $\mathbf{t}(r) = \mathbf{s}(r+1)$, $\mathbf{t}(r+1) = \mathbf{s}(r)$. Let \mathbf{i}_t and \mathbf{i}_s be the residue sequences of \mathbf{t} and \mathbf{s} , respectively. Hence, we have $\mathbf{i}_s = \mathbf{i}_t \cdot s_r$ because $\mathbf{s} = \mathbf{t} \cdot s_r$. By the construction, we have $\prod_{w \in L_{k,\ell}(\mathbf{i}_t)} w(\mathbf{t}) = \prod_{w \in L_{k,\ell}(\mathbf{i}_s)} w(\mathbf{s})$ when $\ell \neq r, r+1$, and

$$\prod_{w \in L_{k,r}(\mathbf{i}_t)} w(\mathbf{t}) = \prod_{w \in L_{k,r+1}(\mathbf{i}_s)} w(\mathbf{s}), \quad \prod_{w \in L_{k,r+1}(\mathbf{i}_t)} w(\mathbf{t}) = \prod_{w \in L_{k,r}(\mathbf{i}_s)} w(\mathbf{s}).$$

As $\mathbf{s} = \mathbf{t} \cdot s_r$ is an up-down tableau, we have $r > 1$. Hence we have $\prod_{w \in S_k(\mathbf{i}_t)} w(\mathbf{t}) = \prod_{w \in S_k(\mathbf{i}_s)} w(\mathbf{s})$. Therefore, by combining the above results, we have $P_k(\mathbf{t}) = P_k(\mathbf{s})$. We can prove $Q_k(\mathbf{t}) = Q_k(\mathbf{s})$ following by the same process. \square

6.14. Lemma. Suppose $1 \leq r, k \leq n-1$ with $r < k-1$ and \mathbf{t} is an up-down tableau with $\mathbf{t}(r) + \mathbf{t}(r+1) = 0$. For any $\mathbf{s} \stackrel{r}{\sim} \mathbf{t}$, we have $P_k(\mathbf{t}) = P_k(\mathbf{s})$ and $Q_k(\mathbf{t}) = Q_k(\mathbf{s})$.

Proof. Because $\mathbf{s} \stackrel{r}{\sim} \mathbf{t}$, we have $\mathbf{t}(\ell) = \mathbf{s}(\ell)$ for $1 \leq \ell \leq n$ and $\ell \neq r, r+1$, and $c_s(r) + c_s(r+1) = c_t(r) + c_t(r+1) = 0$. Let \mathbf{i}_t and \mathbf{i}_s be the residue sequences of \mathbf{t} and \mathbf{s} , respectively. By the construction, when $\ell \neq r, r+1$, we have $\prod_{w \in L_{k,\ell}(\mathbf{i}_t)} w(\mathbf{t}) = \prod_{w \in L_{k,\ell}(\mathbf{i}_s)} w(\mathbf{s})$. As $c_s(r) + c_s(r+1) = c_t(r) + c_t(r+1) = 0$, we have

$$\prod_{w \in L_{k,r}(\mathbf{i}_t)} w(\mathbf{t}) \prod_{w \in L_{k,r+1}(\mathbf{i}_t)} w(\mathbf{t}) = \prod_{w \in L_{k,r}(\mathbf{i}_s)} w(\mathbf{s}) \prod_{w \in L_{k,r+1}(\mathbf{i}_s)} w(\mathbf{s}) = 1.$$

As $c_t(1) = c_s(1)$, we have $\prod_{w \in S_k(\mathbf{i}_t)} w(\mathbf{t}) = \prod_{w \in S_k(\mathbf{i}_s)} w(\mathbf{s})$. Therefore, by combining the above results, we have $P_k(\mathbf{t}) = P_k(\mathbf{s})$. We can prove $Q_k(\mathbf{t}) = Q_k(\mathbf{s})$ following by the same process. \square

The next Lemma is used to prove Lemma 6.16, and also will be used in the next section when we prove the essential commutation relations of $\mathcal{B}_n(\delta)$.

6.15. Lemma. Suppose $\mathbf{i} \in I^n$ with $i_k + i_{k+1} = 0$ for $1 \leq k \leq n-1$ and $h_k(\mathbf{i}) = 0$. Then $(-1)^{a_k(\mathbf{i})} = 1$ when $i_k = 0$ and $(-1)^{a_k(\mathbf{i})+a_k(\mathbf{i} \cdot s_k)} = 1$ when $i_k \neq 0$.

Proof. Suppose $i_k = 0$. We have $a_k(\mathbf{i}) = \#\{1 \leq r \leq k-1 \mid i_r \in \{-1, 1\}\} + 1 + \delta_{\frac{i_k - i_{k+1}}{2}, \frac{\delta-1}{2}}$ by the definition of $a_k(\mathbf{i})$. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and write $\mathbf{t}_{k-1} = \lambda$. By (3.7), we have $|\mathcal{A}\mathcal{R}_\lambda(0)| = 1$. Hence by Corollary 3.17, we have

$$\begin{cases} \#\{1 \leq r \leq k-1 \mid i_r \in \{-1, 1\}\} \text{ is odd if } \frac{\delta-1}{2} \neq 0, \\ \#\{1 \leq r \leq k-1 \mid i_r \in \{-1, 1\}\} \text{ is even if } \frac{\delta-1}{2} = 0, \end{cases}$$

which implies that $a_k(\mathbf{i})$ is even. Hence $(-1)^{a_k(\mathbf{i})} = 1$ when $i_k = 0$.

Suppose $i_k \neq 0$. Because $i_k - i_{k+1} = 2i_k = -2i_{k+1}$, we have

$$\begin{aligned} a_k(\mathbf{i}) &= \#\{1 \leq r \leq k-1 \mid i_r \in \{i_k, i_k - 1, i_{k+1}, i_{k+1} + 1\}\} + \delta_{i_k, (\delta-1)/2}, \\ a_k(\mathbf{i} \cdot s_k) &= \#\{1 \leq r \leq k-1 \mid i_r \in \{i_{k+1}, i_{k+1} - 1, i_k, i_k + 1\}\} + \delta_{i_{k+1}, (\delta-1)/2}. \end{aligned}$$

Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and write $\mathbf{t}_{k-1} = \lambda$. Because $h_k(\mathbf{i}) = 0$, we have $i_k \neq \pm \frac{1}{2}$ by Lemma 3.7. As $i_k = -i_{k+1}$, we have $\{i_k \pm 1\} \cap \{i_{k+1} \pm 1\} = \emptyset$. Hence by the definition of $a_k(\mathbf{i})$ and the construction of λ , we have

$$\begin{aligned} (-1)^{a_k(\mathbf{i})+a_k(\mathbf{i} \cdot s_k)} &= (-1)^{\#\{1 \leq r \leq k-1 \mid i_r \in \{i_k \pm 1, i_{k+1} \pm 1\}\} + \delta_{i_k, (\delta-1)/2} + \delta_{i_{k+1}, (\delta-1)/2}} \\ &= (-1)^{\#\{\alpha \in [\lambda] \mid \text{res}(\alpha) \in \{i_k \pm 1, i_{k+1} \pm 1\}\} + \delta_{i_k, (\delta-1)/2} + \delta_{i_{k+1}, (\delta-1)/2}}. \end{aligned} \quad (6.11)$$

Because $h_k(\mathbf{i}) = 0$ and $i_k \neq 0$, by Lemma 3.8, there exist β and γ such that $\text{res}(\beta) = -\text{res}(\gamma) = i_k$ and either $\beta, \gamma \in \mathcal{A}(\lambda)$ or $\beta, \gamma \in \mathcal{B}(\lambda)$. Therefore, by the construction of λ , we have

$$\begin{cases} \#\{\alpha \in [\lambda] \mid \text{res}(\alpha) \in \{i_m - 1, i_m + 1\}\} \text{ is odd if } i_m \neq \frac{\delta-1}{2}, \\ \#\{\alpha \in [\lambda] \mid \text{res}(\alpha) \in \{i_m - 1, i_m + 1\}\} \text{ is even if } i_m = \frac{\delta-1}{2}, \end{cases}$$

where $m \in \{k, k+1\}$. Therefore, by (6.11) and $i_k \neq i_{k+1}$, we have $(-1)^{a_k(\mathbf{i})+a_k(\mathbf{i} \cdot s_k)} = 1$. \square

6.16. Lemma. Suppose $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. If $i_k = i_{k+1}$ and $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$ for some $1 \leq k \leq n-1$, we have $V_k(\mathbf{t}) = s_k(\mathbf{t})$.

Proof. When $i_k = i_{k+1}$ and $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$, we have $i_k = i_{k+1} = 0$, which implies $\mathbf{i} \in I_{k,0}^n$. By Lemma 6.12 and Lemma 6.15, we have $P_k(\mathbf{t})Q_k(\mathbf{t}) = e_k(\mathbf{t}, \mathbf{t})$. Therefore

$$V_k(\mathbf{t}) = \frac{P_k(\mathbf{t})Q_k(\mathbf{t}) - 1}{c_t(k) - c_t(k+1)} = \frac{e_k(\mathbf{t}, \mathbf{t}) - 1}{c_t(k) + c_t(k)} = s_k(\mathbf{t}, \mathbf{t}),$$

which completes the proof. \square

6.17. Lemma. Suppose $1 \leq k \leq n-1$ and \mathbf{t} is an up-down tableau. For up-down tableau \mathbf{s} , we have $V_k(\mathbf{t}) = V_k(\mathbf{s})$ if one of the following conditions holds:

- (a). If $\mathbf{s} = \mathbf{t} \cdot s_r$ for some $1 \leq r < k-1$.
- (b). If $\mathbf{t}(r) + \mathbf{t}(r+1) = \mathbf{s}(r) + \mathbf{s}(r+1) = 0$ and $\mathbf{t} \stackrel{r}{\sim} \mathbf{s}$ for some $1 \leq r < k-1$.

Proof. Because $r < k-1$, in both (a) and (b), we have $c_t(k) = c_s(k)$ and $c_t(k+1) = c_s(k+1)$. Hence we have $c_t(k) - c_t(k+1) = c_s(k) - c_s(k+1)$. By Lemma 6.13 and Lemma 6.14, we have $P_k(\mathbf{t}) = P_k(\mathbf{s})$ and $Q_k(\mathbf{t}) = Q_k(\mathbf{s})$. Hence, we have

$$V_k(\mathbf{t}) = \frac{P_k(\mathbf{t})Q_k(\mathbf{t}) - 1}{c_t(k) - c_t(k+1)} = \frac{P_k(\mathbf{s})Q_k(\mathbf{s}) - 1}{c_s(k) - c_s(k+1)} = V_k(\mathbf{s}). \quad \square$$

6.3. Generators of $\mathcal{B}_n(\delta)$

In this subsection, we define

$$G_n(\delta) = \{e(\mathbf{i}) \mid \mathbf{i} \in P^n\} \cup \{y_k \mid 1 \leq k \leq n\} \cup \{\psi_k \mid 1 \leq k \leq n-1\} \cup \{\epsilon_k \mid 1 \leq k \leq n-1\}$$

in $\mathcal{B}_n(\delta)$ and show that $G_n(\delta)$ is a generating set of $\mathcal{B}_n(\delta)$.

By Lemma 6.1 we have defined a set $\{e(\mathbf{i}) \mid \mathbf{i} \in P^n\} \subset \mathcal{L}_n$ where $e(\mathbf{i}) \neq 0$ only if $\mathbf{i} \in I^n$, and by (6.1) we have a set of elements $y_k \in \mathcal{L}_n$ for $1 \leq k \leq n$. Recall that we have showed that $\sum_{\mathbf{i} \in I^n} e(\mathbf{i})^\theta = \sum_{\mathbf{i} \in P^n} e(\mathbf{i})^\theta = 1_{\mathbb{F}}$, $\sum_{\mathbf{i} \in I^n} e(\mathbf{i}) = \sum_{\mathbf{i} \in P^n} e(\mathbf{i}) = 1_R$ and y_k is nilpotent for $1 \leq k \leq n$.

It left us to define $\{\psi_r, \epsilon_r \mid 1 \leq r \leq n-1\}$. Suppose ℓ is a positive integer and $0 \leq r \leq \ell$. Define

$$c_r^{(\ell)} = \begin{cases} 0, & \text{if } r = \ell/2, r \neq 0 \text{ and } 4 \mid \ell, \\ 2, & \text{if } k = \ell/2, r \neq 0 \text{ and } 2 \mid \ell \text{ but } 4 \nmid \ell, \\ 1, & \text{otherwise;} \end{cases} \quad (6.12)$$

and $z_\ell = \sum_{r=0}^\ell c_r^{(\ell)}$. We have $z_\ell > 0$ because $c_0^{(\ell)} = 1$ and $c_r^{(\ell)} \geq 0$ for any r and ℓ .

Suppose $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ and $1 \leq k \leq n-1$. If $i_k \neq i_{k+1}$, by Lemma 6.2 we define

$$\frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{i})^\sigma := \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{c_t(k) - c_t(k+1)} \frac{f_{tt}}{\gamma_t} \in \mathcal{L}_n(\mathcal{O}),$$

and $\frac{1}{L_k - L_{k+1}} e(\mathbf{i}) = \frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{i})^\sigma \otimes_{\mathcal{O}} 1_R \in \mathcal{L}_n$.

6.18. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ with $i_k + i_{k+1} \neq 0$. Then we have $e(\mathbf{i})^\sigma e_k^\sigma = e_k^\sigma e(\mathbf{i})^\sigma = 0$ in $\mathcal{B}_n^\sigma(x)$ and $e(\mathbf{i})e_k = e_k e(\mathbf{i}) = 0$ in $\mathcal{B}_n(\delta)$.

Proof. For any $t \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $t(k) + t(k+1) \neq 0$ because $i_k + i_{k+1} \neq 0$. Hence we have $\frac{f_{tt}}{\gamma_t} e_k^\sigma = e_k^\sigma \frac{f_{tt}}{\gamma_t} = 0$ in $\mathcal{B}_n^\sigma(x)$, which implies $e(\mathbf{i})^\sigma e_k^\sigma = e_k^\sigma e(\mathbf{i})^\sigma = 0$. Then we have $e(\mathbf{i})e_k = e(\mathbf{i})^\sigma e_k^\sigma \otimes_{\mathcal{O}} 1_R = 0$ and $e_k e(\mathbf{i}) = e_k^\sigma e(\mathbf{i})^\sigma \otimes_{\mathcal{O}} 1_R = 0$ in $\mathcal{B}_n(\delta)$. \square

Suppose $1 \leq k \leq n-1$ and $\mathbf{i}, \mathbf{j} \in I^n$. Define

$$e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma := \begin{cases} e(\mathbf{i})^\sigma (s_r^\sigma + e(\mathbf{i})^\sigma \frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{j})^\sigma - \frac{1}{i_k + j_k} e_k^\sigma \\ - \frac{1}{i_k + j_k} \sum_{\ell=1}^\infty \left(-\frac{2}{i_k + j_k}\right)^\ell \frac{1}{z_\ell} \left(\sum_{r=0}^\ell c_r^{(\ell)} (L_k^\sigma - i_k)^{\ell-r} e_k^\sigma (L_k^\sigma - j_k)^r\right) e(\mathbf{j})^\sigma, & \text{if } \mathbf{j} \neq \mathbf{i} \cdot s_k, \\ e(\mathbf{i})^\sigma P_k^\sigma(\mathbf{i})^{-1} (s_k^\sigma - V_k^\sigma(\mathbf{i})) Q_k^\sigma(\mathbf{j})^{-1} e(\mathbf{j})^\sigma, & \text{if } \mathbf{j} = \mathbf{i} \cdot s_k; \end{cases}$$

$$e(\mathbf{i})^\sigma \epsilon_k^\sigma e(\mathbf{j})^\sigma := e(\mathbf{i})^\sigma P_k^\sigma(\mathbf{i})^{-1} e_k^\sigma Q_k^\sigma(\mathbf{j})^{-1} e(\mathbf{j})^\sigma.$$

The above definition is well-defined. It is obvious that $e(\mathbf{i})^\sigma \epsilon_k^\sigma e(\mathbf{j})^\sigma$ is well-defined. For $e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma$ with $\mathbf{j} \neq \mathbf{i} \cdot s_k$, by Lemma 6.18, $e(\mathbf{i})^\sigma \epsilon_k^\sigma e(\mathbf{j})^\sigma \neq 0$ implies $i_k + i_{k+1} = j_k + j_{k+1} = 0$. Hence we have $i_k + j_k \neq 0$ because $\mathbf{j} \neq \mathbf{i} \cdot s_k$, which implies $e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma$ with $\mathbf{j} \neq \mathbf{i} \cdot s_k$ is well-defined. For $e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma$ with $\mathbf{j} = \mathbf{i} \cdot s_k$, $V_k(\mathbf{i})$ is only defined when $i_k = i_{k+1}$. But it only exists if $\mathbf{i} = \mathbf{j}$, otherwise $e(\mathbf{i})^\sigma P_k^\sigma(\mathbf{i})^{-1} V_k^\sigma(\mathbf{i}) Q_k^\sigma(\mathbf{j})^{-1} e(\mathbf{j})^\sigma = 0$. When $\mathbf{i} = \mathbf{j}$, because $\mathbf{j} = \mathbf{i} \cdot s_k$, we have $i_k = i_{k+1}$. Therefore $e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma$ with $\mathbf{j} = \mathbf{i} \cdot s_k$ is well-defined.

Then, define

$$\psi_k^\sigma = \sum_{\mathbf{i} \in I^n} \sum_{\mathbf{j} \in I^n} e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma \in \mathcal{B}_n^\sigma(x), \quad \epsilon_k^\sigma = \sum_{\mathbf{i} \in I^n} \sum_{\mathbf{j} \in I^n} e(\mathbf{i})^\sigma \epsilon_k^\sigma e(\mathbf{j})^\sigma \in \mathcal{B}_n^\sigma(x),$$

and for any $\mathbf{i}, \mathbf{j} \in I^n$ and $1 \leq k \leq n-1$, we define $e(\mathbf{i})\psi_k e(\mathbf{j}) = e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma \otimes_{\mathcal{O}} 1_R$ and $e(\mathbf{i})\epsilon_k e(\mathbf{j}) = e(\mathbf{i})^\sigma \epsilon_k^\sigma e(\mathbf{j})^\sigma \otimes_{\mathcal{O}} 1_R$, and

$$\psi_k = \sum_{\mathbf{i} \in I^n} \sum_{\mathbf{j} \in I^n} e(\mathbf{i})\psi_k e(\mathbf{j}) \in \mathcal{B}_n(\delta), \quad \epsilon_k = \sum_{\mathbf{i} \in I^n} \sum_{\mathbf{j} \in I^n} e(\mathbf{i})\epsilon_k e(\mathbf{j}) \in \mathcal{B}_n(\delta).$$

6.19. Remark. By the definitions of $e(\mathbf{i})^\sigma \psi_k^\sigma e(\mathbf{j})^\sigma$ and $e(\mathbf{i})^\sigma \epsilon_k^\sigma e(\mathbf{j})^\sigma$, it is easy to see that it is equivalent to define $e(\mathbf{i})\psi_k e(\mathbf{j})$ and $e(\mathbf{i})\epsilon_k e(\mathbf{j})$ by

$$e(\mathbf{i})\psi_k e(\mathbf{j}) := \begin{cases} e(\mathbf{i})(s_r + e(\mathbf{i}) \frac{1}{L_k - L_{k+1}} e(\mathbf{j}) - \frac{1}{i_k + j_k} e_k \\ - \frac{1}{i_k + j_k} \sum_{\ell=1}^\infty \left(-\frac{2}{i_k + j_k}\right)^\ell \frac{1}{z_\ell} \left(\sum_{r=0}^\ell c_r^{(\ell)} (L_k - i_k)^{\ell-r} e_k (L_k - j_k)^r\right) e(\mathbf{j})), & \text{if } \mathbf{j} \neq \mathbf{i} \cdot s_k, \\ e(\mathbf{i})P_k(\mathbf{i})^{-1} (s_k - V_k(\mathbf{i})) Q_k(\mathbf{j})^{-1} e(\mathbf{j}), & \text{if } \mathbf{j} = \mathbf{i} \cdot s_k. \end{cases}$$

$$e(\mathbf{i})\epsilon_k e(\mathbf{j}) := e(\mathbf{i})P_k(\mathbf{i})^{-1} e_k Q_k(\mathbf{j})^{-1} e(\mathbf{j}).$$

6.20. Proposition. The elements

$$G_n(\delta) = \{e(\mathbf{i}) \mid \mathbf{i} \in P^n\} \cup \{y_k \mid 1 \leq k \leq n\} \cup \{\psi_k \mid 1 \leq k \leq n-1\} \cup \{\epsilon_k \mid 1 \leq k \leq n-1\}$$

generates $\mathcal{B}_n(\delta)$.

Proof. Suppose $S \subseteq \mathcal{B}_n(\delta)$ is generated by $G_n(\delta)$. It is sufficient to prove that $\{s_k, e_k \mid 1 \leq k \leq n-1\}$ is contained in S .

For any $1 \leq k \leq n-1$ and $\mathbf{i}, \mathbf{j} \in I^n$, we have $e(\mathbf{i})e_k e(\mathbf{j}) = e(\mathbf{i})P_k(\mathbf{i})\epsilon_k Q_k(\mathbf{i})e(\mathbf{j}) \in S$. Hence, by $\sum_{\mathbf{i} \in I^n} e(\mathbf{i}) = 1$, we have $e_k = \sum_{\mathbf{i} \in I^n} \sum_{\mathbf{j} \in I^n} e(\mathbf{i})e_k e(\mathbf{j}) \in S$.

For any $1 \leq k \leq n-1$ and $\mathbf{i}, \mathbf{j} \in I^n$, if $\mathbf{j} \neq \mathbf{i} \cdot s_k$, we have

$$e(\mathbf{i})s_k e(\mathbf{j}) = e(\mathbf{i}) \left(\psi_k - \frac{1}{L_k - L_{k+1}} + \frac{1}{i_k + j_k} e_k + \frac{1}{i_k + j_k} \sum_{\ell=1}^\infty \left(-\frac{2}{i_k + j_k}\right)^\ell \frac{1}{z_\ell} \left(\sum_{r=0}^\ell c_r^{(\ell)} (L_k - i_k)^{\ell-r} e_k (L_k - j_k)^r \right) \right) e(\mathbf{j}) \in S;$$

and for $\mathbf{j} = \mathbf{i} \cdot s_k$, we have

$$e(\mathbf{i})s_k e(\mathbf{j}) = e(\mathbf{i})P_k(\mathbf{i})(\psi_k + V_k(\mathbf{i}))Q_k(\mathbf{j})e(\mathbf{j}) \in S,$$

which implies that $s_k = \sum_{\mathbf{i} \in I^n} \sum_{\mathbf{j} \in I^n} e(\mathbf{i})s_k e(\mathbf{j}) \in S$ by $\sum_{\mathbf{i} \in I^n} e(\mathbf{i}) = 1$. Hence the Proposition holds. \square

Reader may notice that the definitions of $e(\mathbf{i})^\mathcal{O} \psi_k^\mathcal{O} e(\mathbf{j})^\mathcal{O}$ and $e(\mathbf{i}) \psi_k e(\mathbf{j})$ when $\mathbf{j} \neq \mathbf{i} \cdot s_k$ are comparatively complicated. In the rest of this subsection, we simplify the definitions of these two elements, and the results we obtained will be used in the next section.

Suppose \mathbf{t} is an up-down tableau with $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$ and write $\lambda = \mathbf{t}_{k-1} = \mathbf{t}_{k+1}$, $\mu = \mathbf{t}_k$ and $\alpha = \lambda \ominus \mu$. We say \mathbf{t} is k -added if $\mu = \lambda \cup \{\alpha\}$, and \mathbf{t} is k -removed if $\mu = \lambda \setminus \{\alpha\}$. Define A_k to be the set of all k -added up-down tableaux and R_k to be the set of all k -removed up-down tableaux.

6.21. Lemma. Suppose ℓ is a positive integer, and $c_r^{(\ell)}$ and z_ℓ are defined as in (6.12). We have $z_\ell > 0$ and $\sum_{r=0}^\ell (-1)^r c_r^{(\ell)} = \sum_{r=0}^\ell (-1)^{\ell-r} c_r^{(\ell)} = 0$.

Proof. It is obvious that $z_\ell > 0$ because $c_r^{(\ell)} \geq 0$ and $c_1^\ell = 1$ for any ℓ . The following diagram gives the values of $(-1)^r c_r$:

$$\begin{array}{rcllclcl} \ell = 1 : & 1 & -1 & & & & \\ \ell = 2 : & 1 & -2 & 1 & & & \\ \ell = 3 : & 1 & -1 & 1 & -1 & & \\ \ell = 4 : & 1 & -1 & 0 & 1 & -1 & \\ \ell = 5 : & 1 & -1 & 1 & -1 & 1 & -1 \\ \ell = 6 : & 1 & -1 & 1 & -2 & 1 & -1 & 1 \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

By direct calculation, we have $\sum_{r=0}^\ell (-1)^r c_r^{(\ell)} = 0$. Similarly, we have $\sum_{r=0}^\ell (-1)^{\ell-r} c_r^{(\ell)} = 0$, and the Lemma follows. \square

For convenience, we set $d = \frac{x-\delta}{2}$.

6.22. Lemma. Suppose \mathbf{t} is an up-down tableau with residue sequence \mathbf{i} and $1 \leq k \leq n-1$. Then

$$y_k^\mathcal{O} f_{\mathbf{t}} = f_{\mathbf{t}} y_k^\mathcal{O} = \begin{cases} d f_{\mathbf{t}}, & \text{if } \mathbf{t} \in A_k, \\ -d f_{\mathbf{t}}, & \text{if } \mathbf{t} \in R_k. \end{cases}$$

Proof. This Lemma is a direct application of the definition of $y_k^\mathcal{O}$. \square

6.23. Lemma. Suppose $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$. For any integer $\ell > 0$, we have

$$\frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell c_r^{(\ell)} (y_k^\mathcal{O})^{\ell-r} e_k^\mathcal{O} (y_k^\mathcal{O})^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = \begin{cases} z_\ell d^\ell \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} e_k^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}}, & \text{if } \mathbf{s}, \mathbf{t} \in A_k, \\ z_\ell (-d)^\ell \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} e_k^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}}, & \text{if } \mathbf{s}, \mathbf{t} \in R_k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose $\mathbf{s}, \mathbf{t} \in A_k$. By Lemma 6.22 we have

$$\frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell c_r^{(\ell)} (y_k^\mathcal{O})^{\ell-r} e_k^\mathcal{O} (y_k^\mathcal{O})^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell c_r^{(\ell)} d^{\ell-r} e_k^\mathcal{O} d^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = z_\ell d^\ell \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} e_k^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}}.$$

Suppose $\mathbf{s}, \mathbf{t} \in R_k$. By Lemma 6.22 we have

$$\frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell c_r^{(\ell)} (y_k^\mathcal{O})^{\ell-r} e_k^\mathcal{O} (y_k^\mathcal{O})^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell c_r^{(\ell)} (-d)^{\ell-r} e_k^\mathcal{O} (-d)^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = z_\ell (-d)^\ell \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} e_k^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}}.$$

Suppose $\mathbf{s} \in R_k$ and $\mathbf{t} \in A_k$. By Lemma 6.22 and Lemma 6.21 we have

$$\frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell c_r^{(\ell)} (y_k^\mathcal{O})^{\ell-r} e_k^\mathcal{O} (y_k^\mathcal{O})^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell (-1)^{\ell-r} c_r^{(\ell)} d^{\ell-r} e_k^\mathcal{O} d^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = 0.$$

For $\mathbf{s} \in A_k$ and $\mathbf{t} \in R_k$, following the same argument we have $\frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \left(\sum_{r=0}^\ell c_r^{(\ell)} (y_k^\mathcal{O})^{\ell-r} e_k^\mathcal{O} (y_k^\mathcal{O})^r \right) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = 0$, which completes the proof. \square

Suppose \mathbf{s}, \mathbf{t} are up-down tableaux with residue sequences $\mathbf{i} = (i_1, \dots, i_n)$ and $\mathbf{j} = (j_1, \dots, j_n)$, respectively, and $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$. Then $s_k(\mathbf{t}, \mathbf{s}) = \frac{1}{c_s(k)+c_t(k)} (e_k(\mathbf{t}, \mathbf{s}) - \delta_{\mathbf{s}, \mathbf{t}})$. One can see that $i_k + i_{k+1} = j_k + j_{k+1} = 0$. If $\mathbf{j} \neq \mathbf{i} \cdot s_k$, i.e. $i_k + j_k \neq 0$, we have $\frac{1}{c_s(k)+c_t(k)} \in \mathcal{O}$. As $\mathcal{O} = R[[x-\delta]]$, Lemma 6.23 gives us a method to express $e(\mathbf{i})^\mathcal{O} s_k^\mathcal{O} e(\mathbf{j})^\mathcal{O}$ using $e(\mathbf{i})^\mathcal{O} e_k^\mathcal{O} e(\mathbf{j})^\mathcal{O}$ and $y_k^\mathcal{O}$'s.

6.24. Lemma. Suppose \mathbf{s}, \mathbf{t} are up-down tableaux with residue sequences \mathbf{i} and \mathbf{j} , respectively, and $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$. If $\mathbf{j} \neq \mathbf{i}_{s_k}$, we have

$$\frac{f_{ss}}{\gamma_s} s_k^\sigma \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \left(\frac{1}{i_k + j_k} (e_k^\sigma + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell \frac{1}{z_\ell} (\sum_{r=0}^{\ell} c_r^{(\ell)} (y_k^\sigma)^{\ell-r} e_k^\sigma (y_k^\sigma)^r)) \right) \frac{f_{tt}}{\gamma_t} - e(\mathbf{i})^\sigma \frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{j})^\sigma \frac{f_{tt}}{\gamma_t}.$$

Proof. As $s_k(\mathbf{t}, \mathbf{s}) = \frac{1}{c_s(k) + c_t(k)} (e_k(\mathbf{t}, \mathbf{s}) - \delta_{s,t})$, we have

$$\frac{f_{ss}}{\gamma_s} s_k^\sigma \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma - \delta_{s,t}}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t} - \frac{f_{ss}}{\gamma_s} \frac{\delta_{s,t}}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t}. \quad (6.13)$$

If $\mathbf{s} = \mathbf{t}$, we have $\mathbf{i} = \mathbf{j}$. Hence, by the definition of $\frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{j})^\sigma$, we have

$$\frac{f_{ss}}{\gamma_s} \frac{\delta_{s,t}}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t} = \frac{\delta_{s,t}}{c_t(k) - c_t(k+1)} \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} e(\mathbf{i})^\sigma \frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{j})^\sigma \frac{f_{tt}}{\gamma_t}. \quad (6.14)$$

By (6.13) and (6.14), it is sufficient to show that

$$\frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \left(\frac{1}{i_k + j_k} (e_k^\sigma + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell \frac{1}{z_\ell} (\sum_{r=0}^{\ell} c_r^{(\ell)} (y_k^\sigma)^{\ell-r} e_k^\sigma (y_k^\sigma)^r)) \right) \frac{f_{tt}}{\gamma_t}. \quad (6.15)$$

Suppose $\mathbf{s}, \mathbf{t} \in A_k$. We have

$$\frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{2d + i_k + j_k} \frac{f_{tt}}{\gamma_t} = \frac{1}{i_k + j_k} \sum_{\ell=0}^{\infty} (-\frac{2}{i_k + j_k})^\ell d^\ell \frac{f_{ss}}{\gamma_s} e_k^\sigma \frac{f_{tt}}{\gamma_t}.$$

Then by Lemma 6.23, we have

$$\begin{aligned} & \frac{f_{ss}}{\gamma_s} \left(\frac{1}{i_k + j_k} (e_k^\sigma + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell \frac{1}{z_\ell} (\sum_{r=0}^{\ell} c_r^{(\ell)} (y_k^\sigma)^{\ell-r} e_k^\sigma (y_k^\sigma)^r)) \right) \frac{f_{tt}}{\gamma_t} \\ &= \frac{1}{i_k + j_k} \left(1 + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell d^\ell \right) \frac{f_{ss}}{\gamma_s} e_k^\sigma \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t}. \end{aligned}$$

Suppose $\mathbf{s}, \mathbf{t} \in R_k$. We have

$$\frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{-2d + i_k + j_k} \frac{f_{tt}}{\gamma_t} = \frac{1}{i_k + j_k} \sum_{\ell=0}^{\infty} (-\frac{2}{i_k + j_k})^\ell (-d)^\ell \frac{f_{ss}}{\gamma_s} e_k^\sigma \frac{f_{tt}}{\gamma_t}.$$

Then by Lemma 6.23, we have

$$\begin{aligned} & \frac{f_{ss}}{\gamma_s} \left(\frac{1}{i_k + j_k} (e_k^\sigma + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell \frac{1}{z_\ell} (\sum_{r=0}^{\ell} c_r^{(\ell)} (y_k^\sigma)^{\ell-r} e_k^\sigma (y_k^\sigma)^r)) \right) \frac{f_{tt}}{\gamma_t} \\ &= \frac{1}{i_k + j_k} \left(1 + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell (-d)^\ell \right) \frac{f_{ss}}{\gamma_s} e_k^\sigma \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t}. \end{aligned}$$

Suppose $\mathbf{s} \in A_k$ and $\mathbf{t} \in R_k$. We have $c_s(k) = d + i_k$ and $c_t(k) = -d + j_k$. Then

$$\frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t} = \frac{1}{i_k + j_k} \frac{f_{ss}}{\gamma_s} e_k^\sigma \frac{f_{tt}}{\gamma_t}.$$

Then by Lemma 6.23, we have

$$\frac{f_{ss}}{\gamma_s} \left(\frac{1}{i_k + j_k} (e_k^\sigma + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell \frac{1}{z_\ell} (\sum_{r=0}^{\ell} c_r^{(\ell)} (y_k^\sigma)^{\ell-r} e_k^\sigma (y_k^\sigma)^r)) \right) \frac{f_{tt}}{\gamma_t} = \frac{1}{i_k + j_k} \frac{f_{ss}}{\gamma_s} e_k^\sigma \frac{f_{tt}}{\gamma_t} = \frac{f_{ss}}{\gamma_s} \frac{e_k^\sigma}{c_s(k) + c_t(k)} \frac{f_{tt}}{\gamma_t}.$$

Follow the same argument, we can show that (6.15) holds for $\mathbf{s} \in R_k$ and $\mathbf{t} \in A_k$. Therefore (6.15) holds, which proves the Lemma. \square

6.25. Lemma. Suppose $1 \leq k \leq n-1$ and \mathbf{s} is an up-down tableau with residue sequence \mathbf{i} and $s(k) + s(k+1) \neq 0$. Then we have

$$\frac{f_{ss}}{\gamma_s} s_k^\sigma \frac{f_{ss}}{\gamma_s} = -\frac{f_{ss}}{\gamma_s} e(\mathbf{i})^\sigma \frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{j})^\sigma \frac{f_{ss}}{\gamma_s}.$$

Proof. By Theorem 2.18, we have $\frac{f_{ss}}{\gamma_s} s_k^\sigma \frac{f_{ss}}{\gamma_s} = \frac{1}{c_s(k+1) - c_s(k)} \frac{f_{ss}}{\gamma_s}$. The Lemma follows because $\frac{f_{ss}}{\gamma_s} e(\mathbf{i})^\sigma \frac{1}{L_k^\sigma - L_{k+1}^\sigma} e(\mathbf{j})^\sigma \frac{f_{ss}}{\gamma_s} = \frac{1}{c_s(k) - c_s(k+1)} \frac{f_{ss}}{\gamma_s}$. \square

By Lemma 6.24 and Lemma 6.25, we can simplify the definition of $e(\mathbf{i})^\theta \psi_k^\theta e(\mathbf{j})^\theta$ and $e(\mathbf{i})\psi_k e(\mathbf{j})$ when $\mathbf{j} \neq \mathbf{i} \cdot s_k$.

6.26. Corollary. *Suppose $\mathbf{i}, \mathbf{j} \in I^n$. We have $e(\mathbf{i})^\theta \psi_k^\theta e(\mathbf{j})^\theta = 0$ and $e(\mathbf{i})\psi_k e(\mathbf{j}) = 0$ if $\mathbf{i} \neq \mathbf{j} \cdot s_k$.*

Proof. Suppose $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})$. If $\mathbf{s}(k) + \mathbf{s}(k+1) \neq 0$, by Theorem 2.18, we have $f_{\mathbf{s}\mathbf{s}} s_k^\theta e(\mathbf{j})^\theta \neq 0$ only if $\mathbf{j} = \mathbf{i}$ or $\mathbf{j} = \mathbf{i} \cdot s_k$. Hence we assume $\mathbf{j} = \mathbf{i}$. Then by Lemma 6.25, we have

$$f_{\mathbf{s}\mathbf{s}} s_k^\theta e(\mathbf{j})^\theta = f_{\mathbf{s}\mathbf{s}} s_k^\theta \frac{f_{\mathbf{s}\mathbf{s}}}{\gamma_{\mathbf{s}}} = -f_{\mathbf{s}\mathbf{s}} e(\mathbf{i})^\theta \frac{1}{L_k^\theta - L_{k+1}^\theta} e(\mathbf{j})^\theta \frac{f_{\mathbf{s}\mathbf{s}}}{\gamma_{\mathbf{s}}}. \quad (6.16)$$

If $\mathbf{s}(k) + \mathbf{s}(k+1) = 0$, by Lemma 6.1 and Lemma 6.24, we have

$$\begin{aligned} f_{\mathbf{s}\mathbf{s}} s_k^\theta e(\mathbf{j})^\theta &= f_{\mathbf{s}\mathbf{s}} s_k^\theta \left(\sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{j})} \frac{f_{\mathbf{t}\mathbf{t}}}{\gamma_{\mathbf{t}}} \right) = \sum_{\substack{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{s} \stackrel{k}{\leftarrow} \mathbf{t}}} f_{\mathbf{s}\mathbf{s}} s_k^\theta \frac{f_{\mathbf{t}\mathbf{t}}}{\gamma_{\mathbf{t}}} \\ &= f_{\mathbf{s}\mathbf{s}} \left(\frac{1}{i_k + j_k} (e_k^\theta + \sum_{\ell=1}^{\infty} (-\frac{2}{i_k + j_k})^\ell \frac{1}{z_\ell} (\sum_{r=0}^{\ell} c_r^{(\ell)} (y_k^\theta)^{\ell-r} e_k^\theta (y_k^\theta)^r)) \right) e(\mathbf{j})^\theta - f_{\mathbf{s}\mathbf{s}} \frac{1}{L_k^\theta - L_{k+1}^\theta} e(\mathbf{j})^\theta. \end{aligned} \quad (6.17)$$

Therefore, as $e(\mathbf{i})^\theta y_k^\theta = e(\mathbf{i})^\theta (L_k^\theta - i_k)$ and $y_k^\theta e(\mathbf{j})^\theta = (L_k^\theta - j_k) e(\mathbf{j})^\theta$, (6.16) - (6.17) implies $e(\mathbf{i})^\theta \psi_k^\theta e(\mathbf{j})^\theta = 0$. By lifting the element of $\mathcal{B}_n^\theta(x)$ into $\mathcal{B}_n(\delta)$, we have $e(\mathbf{i})\psi_k e(\mathbf{j}) = e(\mathbf{i})^\theta \psi_k^\theta e(\mathbf{j})^\theta \otimes_{\mathcal{O}} 1_R = 0$. \square

By Corollary 6.26, we re-write the definitions of $e(\mathbf{i})^\theta \psi_k^\theta e(\mathbf{j})^\theta$ and $e(\mathbf{i})\psi_k e(\mathbf{j})$ as

$$\begin{aligned} e(\mathbf{i})^\theta \psi_k^\theta e(\mathbf{j})^\theta &:= \begin{cases} 0, & \text{if } \mathbf{j} \neq \mathbf{i} \cdot s_k, \\ e(\mathbf{i})^\theta P_k^\theta (\mathbf{i})^{-1} (s_k^\theta - V_k^\theta (\mathbf{i})) Q_k^\theta (\mathbf{j})^{-1} e(\mathbf{j})^\theta, & \text{if } \mathbf{j} = \mathbf{i} \cdot s_k; \end{cases} \\ e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{j})^\theta &:= e(\mathbf{i})^\theta P_k^\theta (\mathbf{i})^{-1} e_k^\theta Q_k^\theta (\mathbf{j})^{-1} e(\mathbf{j})^\theta, \end{aligned}$$

and

$$\begin{aligned} e(\mathbf{i})\psi_k e(\mathbf{j}) &:= \begin{cases} 0, & \text{if } \mathbf{j} \neq \mathbf{i} \cdot s_k, \\ e(\mathbf{i}) P_k (\mathbf{i})^{-1} (s_k - V_k (\mathbf{i})) Q_k (\mathbf{j})^{-1} e(\mathbf{j}), & \text{if } \mathbf{j} = \mathbf{i} \cdot s_k. \end{cases} \\ e(\mathbf{i})\epsilon_k e(\mathbf{j}) &:= e(\mathbf{i}) P_k (\mathbf{i})^{-1} e_k Q_k (\mathbf{j})^{-1} e(\mathbf{j}). \end{aligned}$$

Moreover, Corollary 6.26 forces the elements ψ_k to be the intertwining elements of $\mathcal{B}_n(\delta)$, i.e. for any $\mathbf{i} \in P^n$ we have $e(\mathbf{i})\psi_k = \psi_k e(\mathbf{i} \cdot s_k)$.

7. Grading of Brauer algebras

In this section we are going to prove that the elements of $G_n(\delta)$ in $\mathcal{B}_n(\delta)$ follow the same relations associated in $\mathcal{G}_n(\delta)$, which implies that we can define a surjective homomorphism $\mathcal{G}_n(\delta) \rightarrow \mathcal{B}_n(\delta)$ by sending

$$e(\mathbf{i}) \mapsto e(\mathbf{i}), \quad y_r \mapsto y_r, \quad \psi_k \mapsto \psi_k, \quad \epsilon_k \mapsto \epsilon_k$$

with $\mathbf{i} \in P^n$, $1 \leq r \leq n$ and $1 \leq k \leq n-1$. Throughout of this section we are going to work in $\mathcal{B}_n^\theta(x)$ and $\mathcal{B}_n(\delta)$. So $e(\mathbf{i})$, y_r , ψ_k and ϵ_k we used in this section will be elements in $\mathcal{B}_n(\delta)$.

The goal of this section is to prove that relations (3.8) - (3.33) holds in $\mathcal{B}_n(\delta)$. Note that by Lemma 6.1, we have $e(\mathbf{i}) = 0$ if $\mathbf{i} \notin I^n$. By Corollary 5.29, we have similar property in $\mathcal{G}_n(\delta)$. Hence in this section we assume $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I^n$ except when we prove relation (3.20), because apart from (3.20), when any of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ involved in the relations is not a residue sequence, i.e. any of $e(\mathbf{i})$, $e(\mathbf{j})$, $e(\mathbf{k})$ involved in the relations equals 0, then both sides of the relations will be 0 and there is nothing to prove. In relation (3.20), whenever $\mathbf{i} \notin I^n$, i.e. $e(\mathbf{i}) = 0$, the left hand side of the relation is 0. But it is not obvious that the right hand side of the relation equals 0. Hence when we prove (3.20), we will allow \mathbf{i} chosen from P^n rather than assuming $\mathbf{i} \in I^n$.

7.1. Actions of generators on seminormal forms

Most of the calculations in this section are in $\mathcal{B}_n^\theta(x)$. Hence our first step is to calculate the actions of generators on seminormal forms of $\mathcal{B}_n(\delta)$.

Fix $(\lambda, f) \in \widehat{B}_n$ and $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$. The actions of $e(\mathbf{i})^\theta$ and y_k^θ on $f_{\mathbf{s}\mathbf{t}}$ from right and $f_{\mathbf{t}\mathbf{s}}$ from left can be easily calculated by the definitions.

7.1. Lemma. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with residue sequence $\mathbf{j} = (j_1, \dots, j_n)$ and $1 \leq k \leq n$. Then for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$ and $\mathbf{i} \in P^n$, we have

$$\begin{aligned} f_{\mathbf{st}} e(\mathbf{i})^\mathcal{O} &= \delta_{\mathbf{i}, \mathbf{j}} f_{\mathbf{st}}, & f_{\mathbf{st}} y_k^\mathcal{O} &= (c_t(k) - j_k) f_{\mathbf{st}}, \\ e(\mathbf{i})^\mathcal{O} f_{\mathbf{ts}} &= \delta_{\mathbf{i}, \mathbf{j}} f_{\mathbf{ts}}, & y_k^\mathcal{O} f_{\mathbf{ts}} &= (c_t(k) - j_k) f_{\mathbf{ts}}. \end{aligned}$$

Next we calculate the actions of $\psi_k^\mathcal{O}$ and $\epsilon_k^\mathcal{O}$ on $f_{\mathbf{st}}$ from right and $f_{\mathbf{ts}}$ from left when $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$.

7.2. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with residue sequence $\mathbf{i} = (i_1, \dots, i_n)$ and $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$. For any $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$,

(1) if $\mathbf{t} \cdot s_k$ does not exist, we have $f_{\mathbf{st}} \psi_k^\mathcal{O} = \psi_k^\mathcal{O} f_{\mathbf{ts}} = 0$; and if $\mathbf{u} = \mathbf{t} \cdot s_k$ exists, we have

$$f_{\mathbf{st}} \psi_k^\mathcal{O} = \begin{cases} \frac{1}{c_t(k+1) - c_t(k)} f_{\mathbf{st}} + \frac{s_k(\mathbf{t}, \mathbf{u})}{1 - c_u(k) + c_u(k+1)} f_{\mathbf{su}}, & \text{if } i_k = i_{k+1}, \\ s_k(\mathbf{t}, \mathbf{u}) (c_u(k) - c_u(k+1)) f_{\mathbf{su}}, & \text{if } i_k = i_{k+1} - 1, \\ s_k(\mathbf{t}, \mathbf{u}) \frac{c_u(k) - c_u(k+1)}{1 - c_u(k) + c_u(k+1)} f_{\mathbf{su}}, & \text{otherwise}; \end{cases} \quad (7.1)$$

$$\psi_k^\mathcal{O} f_{\mathbf{ts}} = \begin{cases} \frac{1}{c_t(k+1) - c_t(k)} f_{\mathbf{ts}} + \frac{s_k(\mathbf{t}, \mathbf{u})}{1 - c_t(k) + c_t(k+1)} f_{\mathbf{us}}, & \text{if } i_k = i_{k+1}, \\ s_k(\mathbf{t}, \mathbf{u}) (c_t(k) - c_t(k+1)) f_{\mathbf{us}}, & \text{if } i_k = i_{k+1} + 1, \\ s_k(\mathbf{t}, \mathbf{u}) \frac{c_t(k) - c_t(k+1)}{1 - c_t(k) + c_t(k+1)} f_{\mathbf{us}}, & \text{otherwise}. \end{cases} \quad (7.2)$$

(2) we have $f_{\mathbf{st}} \epsilon_k^\mathcal{O} = \epsilon_k^\mathcal{O} f_{\mathbf{ts}} = 0$.

Proof. (1). Suppose $\mathbf{t} \cdot s_k \notin \mathcal{T}_n^{ud}(\lambda)$. By Corollary 6.26, we have $f_{\mathbf{st}} \psi_k^\mathcal{O} = f_{\mathbf{st}} \psi_k^\mathcal{O} e(\mathbf{i} \cdot s_k)^\mathcal{O}$. Let $\mathbf{j} = \mathbf{i} \cdot s_k$. By Theorem 2.18 and the definition of $\psi_k^\mathcal{O}$, we have $f_{\mathbf{st}} \psi_k^\mathcal{O} e(\mathbf{j})^\mathcal{O} = a f_{\mathbf{st}} e(\mathbf{j})^\mathcal{O}$ for some $a \in R(x)$. Hence, we have $f_{\mathbf{st}} \psi_k^\mathcal{O} e(\mathbf{j})^\mathcal{O} \neq 0$ only if $\mathbf{i} = \mathbf{j}$ by Lemma 7.1.

Because $\mathbf{j} = \mathbf{i} \cdot s_k$, we have $i_k = i_{k+1}$. As $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$, by the construction of \mathbf{t} , we have $i_k = i_{k+1} \neq 0$. By the definition of h_k , we have $h_{k+1}(\mathbf{i}) \geq h_k(\mathbf{i}) + 2$. As $\mathbf{i} \in I^n$, we have $-2 \leq h_k(\mathbf{i}), h_{k+1}(\mathbf{i}) \leq 0$ by Lemma 3.6, which forces $h_k(\mathbf{i}) = -2$. By Lemma 3.9, we have either $\mathbf{t}(k) > 0$ and $\mathbf{t}(k+1) < 0$, or $\mathbf{t}(k) < 0$ and $\mathbf{t}(k+1) > 0$. Therefore, $\mathbf{t} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$ by Lemma 2.6. This implies $f_{\mathbf{st}} \psi_k^\mathcal{O} = 0$ when $\mathbf{t} \cdot s_k \notin \mathcal{T}_n^{ud}(\lambda)$. Following the similar argument, we have $\psi_k^\mathcal{O} f_{\mathbf{ts}} = 0$ when $\mathbf{t} \cdot s_k \notin \mathcal{T}_n^{ud}(\lambda)$.

Suppose $\mathbf{t} \cdot s_k \in \mathcal{T}_n^{ud}(\lambda)$. By Corollary 6.26, we have $f_{\mathbf{st}} \psi_k^\mathcal{O} = f_{\mathbf{st}} \psi_k^\mathcal{O} e(\mathbf{i} \cdot s_k)^\mathcal{O}$. When $i_k = i_{k+1}$, we have $\mathbf{j} = \mathbf{i} = \mathbf{i} \cdot s_k$. By the definition of $\psi_k^\mathcal{O}$, Lemma 6.9 and Lemma 6.8, we have

$$\begin{aligned} f_{\mathbf{st}} \psi_k^\mathcal{O} &= P_k(\mathbf{t})^{-1} Q_k(\mathbf{t})^{-1} \left(\frac{1}{c_t(k+1) - c_t(k)} - V_k(\mathbf{t}) \right) f_{\mathbf{st}} + P_k(\mathbf{t})^{-1} Q_k(\mathbf{u})^{-1} s_k(\mathbf{t}, \mathbf{u}) f_{\mathbf{su}} \\ &= \frac{1}{(c_t(k) - c_t(k+1)) V_k(\mathbf{t}) + 1} \frac{1 + (c_t(k) - c_t(k+1)) V_k(\mathbf{t})}{c_t(k+1) - c_t(k)} f_{\mathbf{st}} + \frac{s_k(\mathbf{t}, \mathbf{u})}{1 - c_u(k) + c_u(k+1)} f_{\mathbf{su}} \\ &= \frac{1}{c_t(k+1) - c_t(k)} f_{\mathbf{st}} + \frac{s_k(\mathbf{t}, \mathbf{u})}{1 - c_u(k) + c_u(k+1)} f_{\mathbf{su}}. \end{aligned}$$

When $i_k = i_{k+1} - 1$, by the definition of $\psi_k^\mathcal{O}$ and Lemma 6.9, we have

$$f_{\mathbf{st}} \psi_k^\mathcal{O} = P_k(\mathbf{t})^{-1} Q_k(\mathbf{u})^{-1} s_k(\mathbf{t}, \mathbf{u}) f_{\mathbf{su}} = s_k(\mathbf{t}, \mathbf{u}) (c_u(k) - c_u(k+1)) f_{\mathbf{su}}.$$

For the other cases, by the definition of $\psi_k^\mathcal{O}$ and Lemma 6.9, we have

$$f_{\mathbf{st}} \psi_k^\mathcal{O} = P_k(\mathbf{t})^{-1} Q_k(\mathbf{u})^{-1} s_k(\mathbf{t}, \mathbf{u}) f_{\mathbf{su}} = s_k(\mathbf{t}, \mathbf{u}) \frac{c_u(k) - c_u(k+1)}{1 - c_u(k) + c_u(k+1)} f_{\mathbf{su}}.$$

Therefore, (7.1) holds. Following similar argument, we can prove (7.2).

(2). By Lemma 6.18 and Lemma 7.1, we have $f_{\mathbf{st}} \epsilon_k^\mathcal{O} = f_{\mathbf{st}} e(\mathbf{i})^\mathcal{O} \epsilon_k^\mathcal{O} = 0$. Similarly, we have $\epsilon_k^\mathcal{O} f_{\mathbf{ts}} = 0$. \square

Notice that the actions of $e(\mathbf{i})^\mathcal{O}$, $\psi_k^\mathcal{O}$ and $y_k^\mathcal{O}$ on $f_{\mathbf{st}}$ from right and $f_{\mathbf{ts}}$ from left with $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$ are the same as in the KLR algebras. See Hu-Mathas [9, Lemma 4.23]. Following the same process as Hu-Mathas [9, Proposition 4.28, 4.29], we have the following Corollary.

7.3. Corollary. Suppose $(\lambda, f) \in \widehat{B}_n$, $\mathbf{t} \in \mathcal{T}_n^{ud}(\lambda)$ with residue sequence $\mathbf{i} = (i_1, \dots, i_n)$ and $1 \leq k \leq n-1$. If $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\lambda)$, we have

$$f_{\mathbf{st}} (\psi_k^\mathcal{O})^2 = \begin{cases} 0, & \text{if } i_k = i_{k+1}, \\ f_{\mathbf{st}} (y_k^\mathcal{O} - y_{k+1}^\mathcal{O}), & \text{if } i_k = i_{k+1} - 1, \\ f_{\mathbf{st}} (y_{k+1}^\mathcal{O} - y_k^\mathcal{O}), & \text{if } i_k = i_{k+1} + 1, \\ f_{\mathbf{st}}, & \text{if } |i_k - i_{k+1}| > 1. \end{cases}$$

Similarly, for $1 \leq k \leq n-2$, if $t(k) + t(k+1) \neq 0$, $t(k) + t(k+2) \neq 0$ and $t(k+1) + t(k+2) \neq 0$, we have

$$f_{st}(\psi_k^\mathcal{O} \psi_{k+1}^\mathcal{O} \psi_k^\mathcal{O} - \psi_{k+1}^\mathcal{O} \psi_k^\mathcal{O} \psi_{k+1}^\mathcal{O}) = \begin{cases} f_{st}, & \text{if } i_{k-1} = i_{k+1} = i_k - 1, \\ -f_{st}, & \text{if } i_{k-1} = i_{k+1} = i_k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Finally we calculate the actions of $\psi_k^\mathcal{O}$ and $\epsilon_k^\mathcal{O}$ on f_{st} from right and f_{ts} from left when $t(k) + t(k+1) = 0$.

7.4. Lemma. Suppose $1 \leq k \leq n-1$ and $t \in \mathcal{T}_n^{ud}(\lambda)$ with residue sequence $\mathbf{i} = (i_1, \dots, i_n)$ and $t(k) + t(k+1) = 0$. For any $s \in \mathcal{T}_n^{ud}(\lambda)$, we have

$$f_{st} \psi_k^\mathcal{O} = \begin{cases} 0, & \text{if } i_k = i_{k+1} = 0 \text{ or } h_k(\mathbf{i}) \neq 0, \\ \frac{1}{c_t(k) + c_u(k)} P_k(t)^{-1} e_k(t, u) Q_k(u)^{-1} f_{su}, & \text{otherwise;} \end{cases} \quad (7.3)$$

$$\psi_k^\mathcal{O} f_{ts} = \begin{cases} 0, & \text{if } i_k = i_{k+1} = 0 \text{ or } h_k(\mathbf{i}) \neq 0, \\ \frac{1}{c_t(k) + c_u(k)} P_k(u)^{-1} e_k(t, u) Q_k(t)^{-1} f_{us}, & \text{otherwise,} \end{cases} \quad (7.4)$$

where u is the unique up-down tableau with residue sequence $\mathbf{i} \cdot s_k$ and $u \stackrel{k}{\sim} t$; and

$$f_{st} \epsilon_k^\mathcal{O} = \sum_{v \stackrel{k}{\sim} t} P_k(t)^{-1} e_k(t, v) Q_k(v)^{-1} f_{sv}, \quad (7.5)$$

$$\epsilon_k^\mathcal{O} f_{ts} = \sum_{v \stackrel{k}{\sim} t} P_k(v)^{-1} e_k(t, v) Q_k(t)^{-1} f_{vs}. \quad (7.6)$$

Proof. (7.5) and (7.6) can be easily derived by the definition of $\epsilon_k^\mathcal{O}$ and Theorem 2.18. For (7.3), by Corollary 6.26 we have $f_{st} \psi_k^\mathcal{O} = f_{st} \psi_k^\mathcal{O} e(\mathbf{i} \cdot s_k)^\mathcal{O}$. Hence, if $h_k(\mathbf{i}) \neq 0$, by Lemma 3.20 we have $\mathbf{i} \cdot s_k \notin I^n$, which implies $f_{st} \psi_k^\mathcal{O} = f_{st} \psi_k^\mathcal{O} e(\mathbf{i} \cdot s_k)^\mathcal{O} = 0$ by Lemma 6.1. If $i_k = i_{k+1} = 0$, by Lemma 6.16 we have $f_{st}(s_k^\mathcal{O} - V_k(\mathbf{i})^\mathcal{O}) = 0$, which implies $f_{st} \psi_k^\mathcal{O} = 0$. Therefore we have proved that $f_{st} \psi_k^\mathcal{O} = 0$ if $i_k = i_{k+1} = 0$ or $h_k(\mathbf{i}) \neq 0$.

For the other cases, by Lemma 3.10 there exists a unique $u \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ such that $u \stackrel{k}{\sim} t$. Because $i_k \neq 0$, we have $\mathbf{i} \neq \mathbf{i} \cdot s_k$, and $t \neq u$. Therefore, we have

$$f_{st} \psi_k^\mathcal{O} = f_{st} \psi_k^\mathcal{O} e(\mathbf{i} \cdot s_k)^\mathcal{O} = P_k(t)^{-1} s_k(t, u) Q_k(u)^{-1} f_{su} = \frac{1}{c_t(k) + c_u(k)} P_k(t)^{-1} e_k(t, u) Q_k(u)^{-1} f_{su}.$$

Hence, (7.3) holds. Following the same argument, (7.4) holds. \square

7.2. Idempotent and (essential) commutation relations

In this subsection we are going to prove the idempotent relations, the commutation relations and the essential commutation relations hold in $\mathcal{B}_n(\delta)$. First we prove the idempotent relations.

7.5. Lemma. Suppose $\mathbf{i} \in I^n$. We have $y_1^{\delta_{i_1}, \frac{\delta-1}{2}} e(\mathbf{i}) = 0$.

Proof. Because $\mathbf{i} \in I^n$, we have $i_1 = \frac{\delta-1}{2}$. Hence by Lemma 6.1 and Lemma 7.1, we have

$$y_1^\mathcal{O} e(\mathbf{i})^\mathcal{O} = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} y_1^\mathcal{O} \frac{f_{tt}}{\gamma_t} = \frac{x-\delta}{2} \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{\gamma_t} f_{tt} = \frac{x-\delta}{2} e(\mathbf{i})^\mathcal{O} \in (x-\delta) \mathcal{B}_n^\mathcal{O}(\delta).$$

Hence we have $y_1^{\delta_{i_1}, \frac{\delta-1}{2}} e(\mathbf{i}) = y_1^\mathcal{O} e(\mathbf{i})^\mathcal{O} \otimes_\mathcal{O} 1_R = 0$. \square

7.6. Proposition. In $\mathcal{B}_n(\delta)$, the idempotent relations hold.

Proof. By Lemma 6.1 we have $\sum_{\mathbf{i} \in I^n} e(\mathbf{i})^\mathcal{O} = \sum_{\mathbf{i} \in I^n} e(\mathbf{i})^\mathcal{O} = 1$ and $e(\mathbf{i})^\mathcal{O} e(\mathbf{j})^\mathcal{O} = \delta_{\mathbf{i}, \mathbf{j}} e(\mathbf{i})^\mathcal{O} \in \mathcal{B}_n^\mathcal{O}(x)$, which implies $\sum_{\mathbf{i} \in I^n} e(\mathbf{i}) = 1$ and $e(\mathbf{i}) e(\mathbf{j}) = \delta_{\mathbf{i}, \mathbf{j}} e(\mathbf{i})$ in $\mathcal{B}_n(\delta)$. By Lemma 6.18, we have $e(\mathbf{i}) e_k = e_k e(\mathbf{i}) = 0$ if $i_k + i_{k+1} \neq 0$. Hence by Lemma 7.5, we complete the proof. \square

Next we prove the commutation relations. First we prove that y_k 's commute with $e(\mathbf{i})$'s.

7.7. Lemma. Suppose $\mathbf{i} \in I^n$ and $1 \leq k \leq n$. We have $y_k e(\mathbf{i}) = e(\mathbf{i}) y_k$.

Proof. By Lemma 6.1 and Lemma 7.1, we have

$$y_k^\mathcal{O} e(\mathbf{i})^\mathcal{O} = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} y_1^\mathcal{O} \frac{f_{tt}}{\gamma_t} = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} c_t(k) \frac{f_{tt}}{\gamma_t} = \sum_{t \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{tt}}{\gamma_t} y_k^\mathcal{O} = e(\mathbf{i})^\mathcal{O} y_k^\mathcal{O} \in \mathcal{B}_n^\mathcal{O}(x),$$

which implies $y_k e(\mathbf{i}) = e(\mathbf{i}) y_k$ by lifting the elements into $\mathcal{B}_n(\delta)$. \square

Hence by Corollary 6.26 and Lemma 7.7, we have shown that (3.9) holds in $\mathcal{B}_n(\delta)$. Now we prove the rest of commutation relations hold in $\mathcal{B}_n(\delta)$ as well.

7.8. Lemma. *For $1 \leq k, r \leq n$ and $1 \leq m \leq n-1$, we have $y_k y_r = y_r y_k$, and if $|k-m| > 1$, we have $y_k \psi_m = \psi_m y_k$ and $y_k \epsilon_m = \epsilon_m y_k$.*

Proof. By the definition of y_k , we have $y_k, y_r \in \mathcal{L}$. Because \mathcal{L} is a commutative subalgebra of $\mathcal{B}_n(\delta)$, we have $y_k y_r = y_r y_k$ for $1 \leq k, r \leq n$.

Suppose $1 \leq k \leq n$ and $1 \leq m \leq n-1$, and $|k-m| > 1$. For any $\mathbf{i} \in I^n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, if $\mathbf{t}(m) + \mathbf{t}(m+1) \neq 0$, by Lemma 7.2, without loss of generality, we have $\psi_m^\mathcal{O} f_{\mathbf{t}} = a f_{\mathbf{t}} + b f_{\mathbf{u}}$, where $a, b \in R(x)$ and $\mathbf{u} = \mathbf{t} \cdot s_m$; and if $\mathbf{t}(m) + \mathbf{t}(m+1) = 0$, by Lemma 7.4 we have $\psi_m^\mathcal{O} f_{\mathbf{t}} = a f_{\mathbf{u}}$, where $a \in R(x)$ and $\mathbf{u} \stackrel{m}{\sim} \mathbf{t}$. In either case, for $|k-m| > 1$, we have $y_k^\mathcal{O} \psi_m^\mathcal{O} f_{\mathbf{t}} = \psi_m^\mathcal{O} f_{\mathbf{t}} y_k^\mathcal{O}$ because $c_u(k) = c_t(k)$, which implies

$$y_k^\mathcal{O} \psi_m^\mathcal{O} = \sum_{\substack{\mathbf{i} \in I^n \\ \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})}} y_k^\mathcal{O} \psi_m^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = \sum_{\substack{\mathbf{i} \in I^n \\ \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})}} \psi_m^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} y_k^\mathcal{O} = \psi_m^\mathcal{O} y_k^\mathcal{O} \in \mathcal{B}_n^\mathcal{O}(x).$$

Hence we have $y_k \psi_m = \psi_m y_k$ by lifting the elements into $\mathcal{B}_n(\delta)$.

Suppose $1 \leq k \leq n$ and $1 \leq m \leq n-1$, and $|k-m| > 1$. For any $\mathbf{i} \in I^n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t}(m) + \mathbf{t}(m+1) = 0$, by Lemma 7.4, we have $\epsilon_k^\mathcal{O} f_{\mathbf{t}} = \sum_{\mathbf{u} \stackrel{m}{\sim} \mathbf{t}} a_u f_{\mathbf{u}}$, where $a_u \in R(x)$. Hence, because $|k-m| > 1$, we have $y_k^\mathcal{O} \epsilon_m^\mathcal{O} f_{\mathbf{t}} = \epsilon_m^\mathcal{O} f_{\mathbf{t}} y_k^\mathcal{O}$ because $c_u(k) = c_t(k)$ for any $\mathbf{u} \stackrel{m}{\sim} \mathbf{t}$, which implies

$$y_k^\mathcal{O} \epsilon_m^\mathcal{O} = \sum_{\substack{\mathbf{i} \in I^n \\ \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})}} y_k^\mathcal{O} \epsilon_m^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = \sum_{\substack{\mathbf{i} \in I^n \\ \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})}} \epsilon_m^\mathcal{O} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} y_k^\mathcal{O} = \epsilon_m^\mathcal{O} y_k^\mathcal{O} \in \mathcal{B}_n^\mathcal{O}(x).$$

Hence we have $y_k \epsilon_m = \epsilon_m y_k$ by lifting the elements into $\mathcal{B}_n(\delta)$. \square

7.9. Lemma. *Suppose $\mathbf{i}, \mathbf{j} \in I^n$ and $1 \leq r, k \leq n-1$. If $|k-r| > 1$, we have*

$$\begin{aligned} e(\mathbf{i}) P_k(\mathbf{i})^{-1} s_r e(\mathbf{j}) &= e(\mathbf{i}) s_r P_k(\mathbf{j})^{-1} e(\mathbf{j}), & e(\mathbf{i}) Q_k(\mathbf{i})^{-1} s_r e(\mathbf{j}) &= e(\mathbf{i}) s_r Q_k(\mathbf{j})^{-1} e(\mathbf{j}), & e(\mathbf{i}) V_k(\mathbf{i}) s_r e(\mathbf{j}) &= e(\mathbf{i}) s_r V_k(\mathbf{j}) e(\mathbf{j}) \\ e(\mathbf{i}) P_k(\mathbf{i})^{-1} e_r e(\mathbf{j}) &= e(\mathbf{i}) e_r P_k(\mathbf{j})^{-1} e(\mathbf{j}), & e(\mathbf{i}) Q_k(\mathbf{i})^{-1} e_r e(\mathbf{j}) &= e(\mathbf{i}) e_r Q_k(\mathbf{j})^{-1} e(\mathbf{j}), & e(\mathbf{i}) V_k(\mathbf{i})^{-1} e_r e(\mathbf{j}) &= e(\mathbf{i}) e_r V_k(\mathbf{j})^{-1} e(\mathbf{j}). \end{aligned}$$

Proof. We only prove $e(\mathbf{i}) P_k(\mathbf{i})^{-1} s_r e(\mathbf{j}) = e(\mathbf{i}) s_r P_k(\mathbf{j})^{-1} e(\mathbf{j})$ here. The rest of the equalities follow by the similar argument, except that we use Lemma 6.17 instead of Lemma 6.13 and Lemma 6.14 when we prove

$$e(\mathbf{i}) V_k(\mathbf{i}) s_r e(\mathbf{j}) = e(\mathbf{i}) s_r V_k(\mathbf{j}) e(\mathbf{j}) \quad \text{and} \quad e(\mathbf{i}) V_k(\mathbf{i})^{-1} e_r e(\mathbf{j}) = e(\mathbf{i}) e_r V_k(\mathbf{j})^{-1} e(\mathbf{j}).$$

Suppose $k > r$. Because $|k-r| > 1$, we have $k-1 > r$. Choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. If $\mathbf{t}(r) + \mathbf{t}(r+1) \neq 0$, by Theorem 2.18, without loss of generality, we can write

$$f_{\mathbf{t}} s_r^\mathcal{O} e(\mathbf{j})^\mathcal{O} = a_s f_{\mathbf{t}} + a_t f_{\mathbf{t}},$$

where $\mathbf{s} = \mathbf{t} \cdot s_r$, and $a_s, a_t \in R(x)$. Note we set $a_s = 0$ if \mathbf{s} is not an up-down tableau. Hence, by Lemma 6.13, we have

$$f_{\mathbf{t}} P_k^\mathcal{O}(\mathbf{i})^{-1} s_r^\mathcal{O} e(\mathbf{j})^\mathcal{O} = a_s P_k(\mathbf{t})^{-1} f_{\mathbf{t}} + a_t P_k(\mathbf{t})^{-1} f_{\mathbf{t}} = a_s f_{\mathbf{t}} P_k(\mathbf{s})^{-1} + a_t f_{\mathbf{t}} P_k(\mathbf{t})^{-1} = f_{\mathbf{t}} s_r^\mathcal{O} P_k^\mathcal{O}(\mathbf{j})^{-1} e(\mathbf{j})^\mathcal{O},$$

which implies

$$f_{\mathbf{t}} P_k^\mathcal{O}(\mathbf{i})^{-1} s_r^\mathcal{O} e(\mathbf{j})^\mathcal{O} = f_{\mathbf{t}} s_r^\mathcal{O} P_k(\mathbf{j})^{-1} e(\mathbf{j})^\mathcal{O}, \quad (7.7)$$

when $\mathbf{t}(r) + \mathbf{t}(r+1) \neq 0$.

If $\mathbf{t}(r) + \mathbf{t}(r+1) = 0$, by Theorem 2.18, we have $f_{\mathbf{t}} s_r^\mathcal{O} e(\mathbf{j})^\mathcal{O} = \sum_{\substack{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{s} \stackrel{r}{\sim} \mathbf{t}}} a_s f_{\mathbf{s}}$, where $a_s \in R(x)$. Hence, by Lemma 6.14, we have

$$f_{\mathbf{t}} P_k^\mathcal{O}(\mathbf{i})^{-1} s_r^\mathcal{O} e(\mathbf{j})^\mathcal{O} = \sum_{\substack{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{s} \stackrel{r}{\sim} \mathbf{t}}} a_s P_k(\mathbf{t})^{-1} f_{\mathbf{s}} = \sum_{\substack{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{s} \stackrel{r}{\sim} \mathbf{t}}} a_s f_{\mathbf{s}} P_k(\mathbf{s})^{-1} = f_{\mathbf{t}} s_r^\mathcal{O} P_k^\mathcal{O}(\mathbf{j})^{-1} e(\mathbf{j})^\mathcal{O},$$

which implies

$$f_{\mathbf{t}} P_k^\mathcal{O}(\mathbf{i})^{-1} s_r^\mathcal{O} e(\mathbf{j})^\mathcal{O} = f_{\mathbf{t}} s_r^\mathcal{O} P_k(\mathbf{j})^{-1} e(\mathbf{j})^\mathcal{O}, \quad (7.8)$$

when $\mathbf{t}(r) + \mathbf{t}(r+1) = 0$.

By (7.7) and (7.8), for arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $f_{\mathbf{t}} P_k^{\mathcal{O}}(\mathbf{i})^{-1} s_r^{\mathcal{O}} e(\mathbf{j})^{\mathcal{O}} = f_{\mathbf{t}} s_r^{\mathcal{O}} P_k(\mathbf{j})^{-1} e(\mathbf{j})^{\mathcal{O}}$. Therefore, by Lemma 6.1, we have

$$\begin{aligned} e(\mathbf{i})^{\mathcal{O}} P_k^{\mathcal{O}}(\mathbf{i})^{-1} s_r^{\mathcal{O}} e(\mathbf{j})^{\mathcal{O}} &= \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} f_{\mathbf{t}} P_k^{\mathcal{O}}(\mathbf{i})^{-1} s_r^{\mathcal{O}} e(\mathbf{j})^{\mathcal{O}} \\ &= \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} f_{\mathbf{t}} s_r^{\mathcal{O}} e(\mathbf{j})^{\mathcal{O}} P_k^{\mathcal{O}}(\mathbf{j})^{-1} = e(\mathbf{i})^{\mathcal{O}} s_r^{\mathcal{O}} P_k(\mathbf{j})^{-1} e(\mathbf{j})^{\mathcal{O}} \in \mathcal{B}_n^{\mathcal{O}}(x). \end{aligned}$$

By lifting the elements into $\mathcal{B}_n(\delta)$, we have $e(\mathbf{i}) P_k(\mathbf{i})^{-1} s_r e(\mathbf{j}) = e(\mathbf{i}) s_r P_k(\mathbf{j})^{-1} e(\mathbf{j})$ when $k > r$.

Suppose $k < r$. Because $|k - r| > 1$, we have $k < r - 1$. Choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. By Theorem 2.18, without loss of generality, we have $f_{\mathbf{t}} s_r^{\mathcal{O}} e(\mathbf{j})^{\mathcal{O}} = \sum_{\substack{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{s}|_{r-1} = \mathbf{t}|_{r-1}}} a_{\mathbf{s}} f_{\mathbf{t}\mathbf{s}}$, where $a_{\mathbf{s}} \in R(x)$. By the definition, $P_k(\mathbf{s})$ depends on $\mathbf{s}(1), \mathbf{s}(2), \dots, \mathbf{s}(k+1)$. Because $k < r - 1$, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$ with $a_{\mathbf{s}} \neq 0$, we have $P_k(\mathbf{s})^{-1} = P_k(\mathbf{t})^{-1}$. Therefore, we have

$$f_{\mathbf{t}} P_k^{\mathcal{O}}(\mathbf{i})^{-1} s_r^{\mathcal{O}} e(\mathbf{j})^{\mathcal{O}} = \sum_{\substack{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{s}|_{r-1} = \mathbf{t}|_{r-1}}} a_{\mathbf{s}} P_k(\mathbf{t}) f_{\mathbf{t}\mathbf{s}} = \sum_{\substack{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{s}|_{r-1} = \mathbf{t}|_{r-1}}} a_{\mathbf{s}} f_{\mathbf{t}\mathbf{s}} P_k(\mathbf{s}) = f_{\mathbf{t}} s_r^{\mathcal{O}} P_k^{\mathcal{O}}(\mathbf{j})^{-1} e(\mathbf{j})^{\mathcal{O}}.$$

Because \mathbf{t} is chosen arbitrary, following the same argument as when $k > r$, we have $e(\mathbf{i}) P_k(\mathbf{i})^{-1} s_r e(\mathbf{j}) = e(\mathbf{i}) s_r P_k(\mathbf{j})^{-1} e(\mathbf{j})$ when $k < r$. \square

7.10. Lemma. For $1 \leq k, r \leq n - 1$ where $|k - r| > 1$, we have $\psi_k \psi_r = \psi_r \psi_k$, $\psi_k \epsilon_r = \epsilon_r \psi_k$ and $\epsilon_k \epsilon_r = \epsilon_r \epsilon_k$.

Proof. Without loss of generality, we assume $k > r$. We will only prove $\psi_k \psi_r = \psi_r \psi_k$, and the rest of the equalities follow by the same argument.

Choose arbitrary $\mathbf{i} \in I^n$. By the definitions, we have $P_k^{\mathcal{O}}(\mathbf{i})^{-1} e(\mathbf{i})^{\mathcal{O}} = P_k^{\mathcal{O}}(\mathbf{i})^{-1}$ and $Q_k^{\mathcal{O}}(\mathbf{i})^{-1} e(\mathbf{i})^{\mathcal{O}} = Q_k^{\mathcal{O}}(\mathbf{i})^{-1}$, which implies $P_k(\mathbf{i})^{-1} e(\mathbf{i}) = P_k(\mathbf{i})^{-1}$ and $Q_k(\mathbf{i})^{-1} e(\mathbf{i}) = Q_k(\mathbf{i})^{-1}$ by lifting the elements into $\mathcal{B}_n(\delta)$. Because $|k - r| > 1$, s_r and s_k commutes. Hence, by Corollary 6.26 and Lemma 7.9, we have

$$\begin{aligned} \psi_k \psi_r e(\mathbf{i}) &= e(\mathbf{i} \cdot s_k s_r) \psi_k e(\mathbf{i} \cdot s_r) \psi_r e(\mathbf{i}) \\ &= e(\mathbf{i} \cdot s_k s_r) P_k(\mathbf{i} \cdot s_k s_r)^{-1} (s_k - V_k(\mathbf{i} \cdot s_r)) Q_k(\mathbf{i} \cdot s_r)^{-1} P_r(\mathbf{i} \cdot s_r)^{-1} (s_r - V_r(\mathbf{i})) Q_r(\mathbf{i})^{-1} e(\mathbf{i}) \\ &= e(\mathbf{i} \cdot s_k s_r) P_r(\mathbf{i} \cdot s_k s_r)^{-1} (s_r - V_r(\mathbf{i} \cdot s_k)) Q_r(\mathbf{i} \cdot s_k)^{-1} P_k(\mathbf{i} \cdot s_k)^{-1} (s_k - V_k(\mathbf{i})) Q_k(\mathbf{i})^{-1} e(\mathbf{i}) \\ &= e(\mathbf{i} \cdot s_k s_r) \psi_r e(\mathbf{i} \cdot s_k) \psi_k e(\mathbf{i}) = \psi_r \psi_k e(\mathbf{i}). \end{aligned}$$

As \mathbf{i} is chosen arbitrary, we have $\psi_k \psi_r = \psi_r \psi_k$, which completes the proof. \square

7.11. Proposition. In $\mathcal{B}_n(\delta)$, the commutation relations hold.

Proof. By Corollary 6.26, we have $e(\mathbf{i}) \psi_k e(\mathbf{j}) = 0$ if $\mathbf{i} \neq \mathbf{j} \cdot s_k$. Hence, we have $e(\mathbf{i}) \psi_k = e(\mathbf{i}) \psi_k e(\mathbf{i} \cdot s_k) = \psi_k e(\mathbf{i} \cdot s_k)$. Therefore, (3.9) holds by Lemma 7.7. The relation (3.10) holds by Lemma 7.8 and (3.11) holds by Lemma 7.10. \square

In the rest of this subsection, we prove that the essential commutation relations hold in $\mathcal{B}_n(\delta)$. First we introduce the following results, which will be used.

7.12. Lemma. Suppose $1 \leq k \leq n - 1$ and $\mathbf{i} \in I_{k,0}^n$ with $i_k = -i_{k+1} \neq \pm \frac{1}{2}$. Then we have $e(\mathbf{i}) \epsilon_k e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})} e(\mathbf{i})$.

Proof. As $\mathbf{i} \in I_{k,0}^n$ with $i_k = -i_{k+1} \neq \pm \frac{1}{2}$, when $i_k = -i_{k+1} \neq 0$, we have $h_k(\mathbf{i}) = -1$, which implies $h_{k+1}(\mathbf{i}) = -1$. Choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and let $\mathbf{t}_{k-1} = \lambda$ and $\mathbf{t}_k = \mu$. We have a unique $\alpha \in \mathcal{A} \mathcal{R}_{\lambda}(i_k)$ and unique $\beta \in \mathcal{A} \mathcal{R}_{\mu}(i_{k+1})$ because $h_k(\mathbf{i}) = h_{k+1}(\mathbf{i}) = -1$. Therefore, we have $\alpha = \beta$. Similarly, when $i_k = -i_{k+1} = 0$, choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$.

Choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. By Lemma 6.12, we have

$$\frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} \epsilon_k \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} P_k^{\mathcal{O}}(\mathbf{i})^{-1} e_k^{\mathcal{O}} Q_k^{\mathcal{O}}(\mathbf{i})^{-1} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = P_k(\mathbf{t})^{-1} Q_k(\mathbf{t})^{-1} e_k(\mathbf{t}, \mathbf{t}) \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} = (-1)^{a_k(\mathbf{i})} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}}.$$

As $\mathbf{i} \in I_{k,0}^n$ with $i_k = -i_{k+1} \neq \pm \frac{1}{2}$, when $i_k = -i_{k+1} \neq 0$, we have $h_k(\mathbf{i}) = -1$. Hence by Lemma 3.10, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$, we have $\mathbf{s} = \mathbf{t}$. When $i_k = -i_{k+1} = 0$, we have $h_k(\mathbf{i}) = 0$. Hence by Lemma 3.10, there exists a unique up-down tableau $\mathbf{s} \in \mathbf{i} \cdot s_k$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$. Because $\mathbf{i} \cdot s_k = \mathbf{i}$ and $\mathbf{t} \stackrel{k}{\sim} \mathbf{t}$, it forces $\mathbf{s} = \mathbf{t}$. Therefore, we conclude that for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$, we have $\mathbf{s} = \mathbf{t}$. Hence,

$$e(\mathbf{i})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} = \left(\sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{s}}}{\gamma_{\mathbf{s}}} \right) \epsilon_k^{\mathcal{O}} \left(\sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{s}}}{\gamma_{\mathbf{s}}} \right) = \sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{s}}}{\gamma_{\mathbf{s}}} \epsilon_k^{\mathcal{O}} \frac{f_{\mathbf{s}}}{\gamma_{\mathbf{s}}} = (-1)^{a_k(\mathbf{i})} \sum_{\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{s}}}{\gamma_{\mathbf{s}}} = (-1)^{a_k(\mathbf{i})} e(\mathbf{i})^{\mathcal{O}} \in \mathcal{B}_n^{\mathcal{O}}(x),$$

and we have $e(\mathbf{i})\epsilon_k e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})} e(\mathbf{i})$ by lifting elements to $\mathcal{B}_n(\delta)$. \square

7.13. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in I^n$ with $i_k = i_{k+1} = 0$. Then we have $e(\mathbf{i})^\mathcal{O} \psi_k^\mathcal{O} e(\mathbf{i})^\mathcal{O} = 0$ and $e(\mathbf{i})\psi_k e(\mathbf{i}) = 0$.

Proof. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. Because $i_k = i_{k+1} = 0$, we have $c_t(k) + c_t(k+1) = 0$. By Lemma 6.16, we have $V_k(\mathbf{t}) = s_k(\mathbf{t}, \mathbf{t})$. Hence we have

$$\frac{f_{\mathbf{tt}}}{\gamma_{\mathbf{t}}} \psi_k^\mathcal{O} \frac{f_{\mathbf{tt}}}{\gamma_{\mathbf{t}}} = P_k(\mathbf{t}) Q_k(\mathbf{t})(s_k(\mathbf{t}, \mathbf{t}) - V_k(\mathbf{t})) \frac{f_{\mathbf{tt}}}{\gamma_{\mathbf{t}}} = 0.$$

Because $i_k = 0$, we have $h_k(\mathbf{i}) = 0$. By Lemma 3.10, there exists a unique $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$. Because $\mathbf{i} = \mathbf{i} \cdot s_k$, we have $\mathbf{s} = \mathbf{t}$. Therefore,

$$e(\mathbf{i})^\mathcal{O} \psi_k^\mathcal{O} e(\mathbf{i})^\mathcal{O} = \left(\sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{tt}}}{\gamma_{\mathbf{t}}} \right) \psi_k^\mathcal{O} \left(\sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{tt}}}{\gamma_{\mathbf{t}}} \right) = \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{f_{\mathbf{tt}}}{\gamma_{\mathbf{t}}} \psi_k^\mathcal{O} \frac{f_{\mathbf{tt}}}{\gamma_{\mathbf{t}}} = 0,$$

which implies $e(\mathbf{i})\psi_k e(\mathbf{i}) = 0$ by lifting the elements into $\mathcal{B}_n(\delta)$. \square

Recall that $L_k s_k - s_k L_{k+1} = s_k L_k - L_{k+1} s_k = e_k - 1$. The next Proposition shows that the essential commutation relations hold in $\mathcal{B}_n(\delta)$.

7.14. Proposition. In $\mathcal{B}_n(\delta)$, the essential commutation relations hold.

Proof. Suppose $i_k = i_{k+1} = 0$. We have $\mathbf{i} \cdot s_k = \mathbf{i}$. Then by Lemma 6.15, Lemma 7.12 and Lemma 7.13, we have

$$e(\mathbf{i})y_k \psi_k = 0 = e(\mathbf{i})\psi_k y_{k+1} + e(\mathbf{i})\epsilon_k e(\mathbf{i}) - e(\mathbf{i}).$$

Suppose $i_k = i_{k+1} \neq 0$. Then we have $\mathbf{i} \cdot s_k = \mathbf{i}$. Therefore, by Lemma 6.8, we have

$$\begin{aligned} e(\mathbf{i})y_k \psi_k &= e(\mathbf{i})P_k(\mathbf{i})^{-1}(s_k - V_k(\mathbf{i}))Q_k(\mathbf{i})^{-1}y_{k+1} + e(\mathbf{i})\epsilon_k e(\mathbf{i}) - e(\mathbf{i})P_k(\mathbf{i})^{-1}Q_k(\mathbf{i})^{-1}(V_k(\mathbf{i})(L_k - L_{k+1}) + 1) \\ &= e(\mathbf{i})\psi_k y_{k+1} + e(\mathbf{i})\epsilon_k e(\mathbf{i}) - e(\mathbf{i}). \end{aligned}$$

Suppose $i_k \neq i_{k+1}$ and let $\mathbf{j} = \mathbf{i} \cdot s_k$. Notice that $\mathbf{j} \neq \mathbf{i}$. Hence we have $e(\mathbf{i})y_k \psi_k = e(\mathbf{i})\psi_k y_{k+1} + e(\mathbf{i})\epsilon_k e(\mathbf{j})$ by direct calculation. Hence the relation (3.12) holds. By applying the same method as above, (3.13) holds, which completes the proof. \square

7.3. Inverse relations, essential idempotent relations and untwist relations

In this subsection, we are going to prove the inverse relations, essential idempotent relations and untwist relations hold in $\mathcal{B}_n(\delta)$. First we prove the inverse relations of $\mathcal{B}_n(\delta)$.

7.15. Lemma. Suppose $\mathbf{i} \in I^n$ with $|i_k - i_{k+1}| > 1$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $t(k) + t(k+1) = 0$ for $1 \leq k \leq n-1$. Then we have $f_{\mathbf{tt}}(\psi_k^\mathcal{O})^2 = f_{\mathbf{tt}}$ if $h_k(\mathbf{i}) = 0$.

Proof. We have $i_k + i_{k+1} = 0$ because $t(k) + t(k+1) = 0$. Then, as $|i_k - i_{k+1}| > 1$ and $h_k(\mathbf{i}) = 0$, we have $\mathbf{i} \in I_{k,+}^n$.

By Lemma 3.10, there exists a unique $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ such that $\mathbf{u} \stackrel{k}{\sim} \mathbf{t}$ and $c_t(k) - i_k = c_u(k) - i_{k+1}$. Therefore,

$$\begin{aligned} f_{\mathbf{tt}}(\psi_k^\mathcal{O})^2 &= \frac{P_k(\mathbf{t})^{-1}e_k(\mathbf{t}, \mathbf{u})Q_k(\mathbf{u})^{-1}P_k(\mathbf{u})^{-1}e_k(\mathbf{u}, \mathbf{t})Q_k(\mathbf{t})^{-1}}{(c_t(k) + c_u(k))^2} f_{\mathbf{tt}} \\ &= \frac{P_k(\mathbf{t})^{-1}Q_k(\mathbf{t})^{-1}Q_k(\mathbf{u})^{-1}P_k(\mathbf{u})^{-1}e_k(\mathbf{t}, \mathbf{t})e_k(\mathbf{u}, \mathbf{u})}{4(c_t(k) - i_k)(c_u(k) - i_{k+1})} f_{\mathbf{tt}} = (-1)^{a_k(\mathbf{i}) + a_k(\mathbf{i} \cdot s_k)} f_{\mathbf{tt}}, \end{aligned}$$

by Theorem 2.18, Lemma 7.4 and Lemma 6.12. As $h_k(\mathbf{i}) = 0$, by Lemma 6.15, we have $(-1)^{a_k(\mathbf{i}) + a_k(\mathbf{i} \cdot s_k)} = 1$, which proves the Lemma. \square

7.16. Lemma. Suppose $\mathbf{i} \in I^n$ and $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. For any $1 \leq k \leq n-1$, we have

$$f_{\mathbf{tt}}(\psi_k^\mathcal{O})^2 = \begin{cases} 0, & \text{if } i_k = i_{k+1}, \text{ or } i_k + i_{k+1} = 0 \text{ and } h_k(\mathbf{i}) \neq 0, \\ f_{\mathbf{tt}}(y_k^\mathcal{O} - y_{k+1}^\mathcal{O}), & \text{if } i_k = i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0, \\ f_{\mathbf{tt}}(y_{k+1}^\mathcal{O} - y_k^\mathcal{O}), & \text{if } i_k = i_{k+1} + 1 \text{ and } i_k + i_{k+1} \neq 0, \\ f_{\mathbf{tt}}, & \text{otherwise.} \end{cases}$$

Proof. We prove the Lemma by considering each case.

Case 1: $i_k = i_{k+1}$, or $i_k + i_{k+1} = 0$ and $h_k(\mathbf{i}) \neq 0$.

In this case, when $i_k = i_{k+1} \neq 0$, we have $i_k + i_{k+1} \neq 0$, which implies $t(k) + t(k+1) \neq 0$. Hence we have $f_{tt}(\psi_k^\mathcal{O})^2 = 0$ by Corollary 7.3. When $i_k = i_{k+1} = 0$, by Lemma 7.13, we have $e(\mathbf{i})^\mathcal{O} \psi_k^\mathcal{O} = 0$, which implies $f_{tt}(\psi_k^\mathcal{O})^2 = 0$. When $i_k + i_{k+1} = 0$ and $h_k(\mathbf{i}) \neq 0$, we have $\mathbf{i} \cdot s_k \notin I^n$ by Lemma 3.20, which implies $f_{tt}(\psi_k^\mathcal{O})^2 = f_{tt} \psi_k^\mathcal{O} e(\mathbf{i} \cdot s_k)^\mathcal{O} \psi_k^\mathcal{O} = 0$ because $e(\mathbf{i})^\mathcal{O} \psi_k^\mathcal{O} = \psi_k^\mathcal{O} e(\mathbf{i} \cdot s_k)^\mathcal{O}$ by Corollary 6.26 and $e(\mathbf{i} \cdot s_k)^\mathcal{O} = 0$ by Lemma 6.1.

Case 2: $i_k = i_{k+1} - 1$ and $i_k + i_{k+1} \neq 0$.

In this case, $i_k + i_{k+1} \neq 0$ forces $t(k) + t(k+1) \neq 0$. Hence we have $f_{tt}(\psi_k^\mathcal{O})^2 = f_{tt}(y_k^\mathcal{O} - y_{k+1}^\mathcal{O})$ by Corollary 7.3.

Case 3: $i_k = i_{k+1} + 1$ and $i_k + i_{k+1} \neq 0$.

Following the same argument as in Case 2, we have $f_{tt}(\psi_k^\mathcal{O})^2 = f_{tt}(y_{k+1}^\mathcal{O} - y_k^\mathcal{O})$.

Case 4: $|i_k - i_{k+1}| > 1$ with either $i_k + i_{k+1} \neq 0$, or $i_k + i_{k+1} = 0$ and $h_k(\mathbf{i}) = 0$.

First we show this case contains all \mathbf{i} which does not satisfy Case 1 - 3. Choose arbitrary $\mathbf{i} \in I^n$ does not satisfy Case 1 - 3. We have $i_k \neq i_{k+1}$ because \mathbf{i} does not satisfy Case 1. Assume $i_k \neq i_{k+1} \pm 1$. Then $i_k + i_{k+1} = 0$ because \mathbf{i} does not satisfy Case 2 - 3, and if $i_k + i_{k+1} = 0$, we have $h_k(\mathbf{i}) \neq 0$ by Lemma 3.7, which contradicts that \mathbf{i} does not satisfy Case 1. Hence we always have $|i_k - i_{k+1}| > 1$. It is easy to see we have either $i_k + i_{k+1} \neq 0$, or $i_k + i_{k+1} = 0$ and $h_k(\mathbf{i}) = 0$ because \mathbf{i} does not satisfy Case 1. This proves that Case 4 contains all \mathbf{i} which does not satisfy Case 1 - 3.

We separate this case further by considering $t(k) + t(k+1) \neq 0$ and $t(k) + t(k+1) = 0$. When $t(k) + t(k+1) \neq 0$, as $|i_k - i_{k+1}| > 1$, we have $f_{tt}(\psi_k^\mathcal{O})^2 = f_{tt}$ by Corollary 7.3. When $t(k) + t(k+1) = 0$, it forces $i_k + i_{k+1} = 0$. Hence we have $h_k(\mathbf{i}) = 0$ and we have $f_{tt}(\psi_k^\mathcal{O})^2 = f_{tt}$ by Lemma 7.15. \square

7.17. Proposition. In $\mathcal{B}_n(\delta)$, the inverse relations hold.

Proof. In Lemma 7.16, as t is chosen arbitrary, we have

$$e(\mathbf{i})^\mathcal{O} (\psi_k^\mathcal{O})^2 = \begin{cases} 0, & \text{if } i_k = i_{k+1} \text{ or } i_k + i_{k+1} = 0 \text{ and } h_k(\mathbf{i}) \neq 0, \\ e(\mathbf{i})^\mathcal{O} (y_k^\mathcal{O} - y_{k+1}^\mathcal{O}), & \text{if } i_k = i_{k+1} + 1 \text{ and } i_k + i_{k+1} \neq 0, \\ e(\mathbf{i})^\mathcal{O} (y_{k+1}^\mathcal{O} - y_k^\mathcal{O}), & \text{if } i_k = i_{k+1} - 1 \text{ and } i_k + i_{k+1} \neq 0, \\ e(\mathbf{i})^\mathcal{O}, & \text{otherwise,} \end{cases}$$

by Lemma 6.1, and the Proposition follows by lifting the elements into $\mathcal{B}_n(\delta)$. \square

Then we prove the essential idempotent relations. Recall that Lemma 7.12 proved the first relation of (3.15). The next two Lemmas prove the rest of relations of (3.15).

7.18. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in I_{k,+}^n$ with $i_k = -i_{k+1} \neq -\frac{1}{2}$. Then we have

$$e(\mathbf{i}) \epsilon_k e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})+1} (y_{k+1} - y_k) e(\mathbf{i}).$$

Proof. Choose arbitrary $t \in \mathcal{T}_n^{ud}(\mathbf{i})$. As $\mathbf{i} \in I_{k,+}^n$ and $i_k \neq -\frac{1}{2}$, we have $h_k(\mathbf{i}) = 0$ and $i_k \neq 0$. By (3.2) we have $h_{k+1}(\mathbf{i}) = -2$. Hence, by Lemma 3.11, we have either $t(k) + t(k+1) = 0$, or $t(k) + t(k+1) \neq 0$ and $c_t(k) - i_k = c_t(k+1) - i_{k+1}$.

Suppose $t(k) + t(k+1) \neq 0$ and $c_t(k) - i_k = c_t(k+1) - i_{k+1}$. We have

$$f_{tt} \epsilon_k^\mathcal{O} e(\mathbf{i})^\mathcal{O} = 0 = (-1)^{a_k(\mathbf{i})+1} ((c_t(k+1) - i_{k+1}) - (c_t(k) - i_k)) f_{tt} = (-1)^{a_k(\mathbf{i})+1} (y_{k+1}^\mathcal{O} - y_k^\mathcal{O}) f_{tt},$$

by Lemma 7.1 and Lemma 7.2.

Suppose $t(k) + t(k+1) = 0$. We have $c_t(k) - i_k = -(c_t(k+1) - i_{k+1})$. By Lemma 3.10, for any $s \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $s \stackrel{k}{\sim} t$, we have $s = t$. Therefore, by Lemma 6.12, we have

$$f_{tt} \epsilon_k^\mathcal{O} e(\mathbf{i})^\mathcal{O} = f_{tt} P_k(t)^{-1} e_k(t, t) Q_k(t)^{-1} \frac{f_{tt}}{\gamma_t} = (-1)^{a_k(\mathbf{i})} 2(c_t(k) - i_k) f_{tt} = (-1)^{a_k(\mathbf{i})} (y_{k+1}^\mathcal{O} - y_k^\mathcal{O}) f_{tt}.$$

As t is chosen arbitrary, we have $e(\mathbf{i})^\mathcal{O} \epsilon_k^\mathcal{O} e(\mathbf{i})^\mathcal{O} = (-1)^{a_k(\mathbf{i})} (y_{k+1}^\mathcal{O} - y_k^\mathcal{O}) e(\mathbf{i})^\mathcal{O} \in \mathcal{B}_n^\mathcal{O}(x)$ by Lemma 6.1, which proves the Lemma by lifting the elements into $\mathcal{B}_n(\delta)$ and $(y_k + y_{k+1}) \epsilon_k = 0$. \square

7.19. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in I_{k,+}^n$ with $i_k = -i_{k+1} = -\frac{1}{2}$. Then we have

$$e(\mathbf{i}) \epsilon_k e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})+1} (y_{k+1} - y_k) e(\mathbf{i}).$$

Proof. Choose arbitrary $t \in \mathcal{T}_n^{ud}(\mathbf{i})$. As $\mathbf{i} \in I_{k,+}^n$ and $i_k = -\frac{1}{2}$, we have $h_k(\mathbf{i}) = -1$. By (3.2) we have $h_{k+1}(\mathbf{i}) = -2$. Hence, by Lemma 3.11, we have either $t(k) + t(k+1) = 0$, or $t(k) + t(k+1) \neq 0$ and $c_t(k) - i_k = c_t(k+1) - i_{k+1}$. Hence, following the same argument as in the proof of Lemma 7.19, the Lemma holds. \square

Next we prove relations (3.17) - (3.19).

7.20. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in I_{k,0}^n$ with $i_k = -i_{k+1} = \frac{1}{2}$. Then we have

$$y_{k+1}e(\mathbf{i}) = y_k e(\mathbf{i}) - 2y_k e(\mathbf{i})\epsilon_k e(\mathbf{i}) = y_k e(\mathbf{i}) - 2e(\mathbf{i})\epsilon_k e(\mathbf{i})y_k.$$

Proof. Choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. As $\mathbf{i} \in I_{k,0}^n$ and $i_k = \frac{1}{2}$, we have $h_k(\mathbf{i}) = -1$. By (3.2) we have $h_{k+1}(\mathbf{i}) = -2$. Hence, by Lemma 3.11, we have either $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$, or $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$ and $c_t(k) - i_k = c_t(k+1) - i_{k+1}$.

Suppose $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$ and $c_t(k) - i_k = c_t(k+1) - i_{k+1}$. By Lemma 7.1 and Lemma 7.2, we have

$$f_{\mathbf{t}}y_{k+1}^\mathcal{O} = f_{\mathbf{t}}y_k^\mathcal{O} = f_{\mathbf{t}}y_k^\mathcal{O} - 2y_k^\mathcal{O}f_{\mathbf{t}}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O}. \quad (7.9)$$

Suppose $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$. We have $c_t(k) - i_k = -(c_t(k+1) - i_{k+1})$. As $h_k(\mathbf{i}) = -1$, by Lemma 3.10, if $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$, then $\mathbf{s} = \mathbf{t}$. Hence, by Lemma 6.12 and Lemma 7.4, we have

$$f_{\mathbf{t}}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} = P_k(\mathbf{t})^{-1}Q_k(\mathbf{t})^{-1}e_k(\mathbf{t}, \mathbf{t})f_{\mathbf{t}} = (-1)^{a_k(\mathbf{i})}f_{\mathbf{t}}.$$

Therefore, by Lemma 7.1, we have

$$\begin{aligned} f_{\mathbf{t}}(y_{k+1}^\mathcal{O} - y_k^\mathcal{O}) &= ((c_t(k+1) - i_{k+1}) - (c_t(k) - i_k))f_{\mathbf{t}} = -2(c_t(k) - i_k)f_{\mathbf{t}} \\ &= -2f_{\mathbf{t}}y_k^\mathcal{O} = -2(-1)^{a_k(\mathbf{i})}f_{\mathbf{t}}y_k^\mathcal{O}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O}. \end{aligned} \quad (7.10)$$

Hence, by Lemma 6.1, (7.9) and (7.10) implies

$$e(\mathbf{i})^\mathcal{O}y_{k+1}^\mathcal{O} = e(\mathbf{i})^\mathcal{O}y_k^\mathcal{O} - 2y_k^\mathcal{O}e(\mathbf{i})^\mathcal{O}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} \in \mathcal{B}_n^\mathcal{O}(x),$$

and we have $e(\mathbf{i})^\mathcal{O}y_{k+1}^\mathcal{O} = e(\mathbf{i})^\mathcal{O}y_k^\mathcal{O} - 2e(\mathbf{i})^\mathcal{O}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O}y_k^\mathcal{O}$ following the same argument. The Lemma follows by lifting the elements into $\mathcal{B}_n(\delta)$. \square

7.21. Lemma. Suppose $2 \leq k \leq n-1$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = i_{k+1}$. We have $(-1)^{a_k(\mathbf{i})} = (-1)^{a_{k-1}(\mathbf{i})+1}$ if $\mathbf{i} \in I_{k,-}^n \cup I_{k,+}^n$ and $(-1)^{a_k(\mathbf{i})} = (-1)^{a_{k-1}(\mathbf{i})}$ if $\mathbf{i} \in I_{k,0}^n$.

Proof. By Lemma 4.17, we have $\deg_{k-1}(\mathbf{i}) = -\deg_k(\mathbf{i})$. Hence $\mathbf{i} \in I_{k-1,0}^n$ if and only if $\mathbf{i} \in I_{k,0}^n$, $\mathbf{i} \in I_{k-1,-}^n$ if and only if $\mathbf{i} \in I_{k,+}^n$ and $\mathbf{i} \in I_{k-1,+}^n$ if and only if $\mathbf{i} \in I_{k,-}^n$.

Choose any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $c_t(k-1) = -c_t(k) = c_t(k+1)$. Because $i_{k-1} = -i_k = i_{k+1}$, such \mathbf{t} exists. By $e_k^\mathcal{O} = e_k^\mathcal{O}e_{k-1}^\mathcal{O}e_k^\mathcal{O}$, we have $e_k(\mathbf{t}, \mathbf{t}) = e_k(\mathbf{t}, \mathbf{t})e_{k-1}(\mathbf{t}, \mathbf{t})e_k(\mathbf{t}, \mathbf{t})$, which implies $e_k(\mathbf{t}, \mathbf{t})e_{k-1}(\mathbf{t}, \mathbf{t}) = 1$. By Lemma 6.10 and Lemma 6.12, we have

$$1 = P_k(\mathbf{t})Q_{k-1}(\mathbf{t})Q_k(\mathbf{t})P_{k-1}(\mathbf{t}) = P_k(\mathbf{t})Q_k(\mathbf{t})P_{k-1}(\mathbf{t})Q_{k-1}(\mathbf{t}) = \begin{cases} -(-1)^{a_k(\mathbf{i})}(-1)^{a_{k-1}(\mathbf{i})}, & \text{if } \mathbf{i} \in I_{k,-}^n \cup I_{k,+}^n, \\ (-1)^{a_k(\mathbf{i})}(-1)^{a_{k-1}(\mathbf{i})}, & \text{if } \mathbf{i} \in I_{k,0}^n, \end{cases}$$

which proves the Lemma. \square

7.22. Lemma. Suppose $2 \leq k \leq n-1$ and $\mathbf{i} \in I_{k,0}^n$ with $i_{k-1} = -i_k = i_{k+1} = \frac{1}{2}$. Then we have

$$e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})}e(\mathbf{i})\epsilon_k e(\mathbf{i}) + 2(-1)^{a_{k-1}(\mathbf{i})}e(\mathbf{i})\epsilon_{k-1}e(\mathbf{i}) - e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}) - e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}).$$

Proof. Because $\mathbf{i} \in I_{k,0}^n$ and $i_k = -\frac{1}{2}$, we have $h_k(\mathbf{i}) = -2$ and $h_{k+1}(\mathbf{i}) = h_{k-1}(\mathbf{i}) = -h_k(\mathbf{i}) - 3 = -1$. As $h_{k-1}(\mathbf{i}) = -1$ and $i_{k-1} = \frac{1}{2}$, we have $\mathbf{i} \in I_{k-1,0}^n$.

For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, write $\lambda = \mathbf{t}_{k-1}$ and $\mu = \mathbf{t}_k$. Let $\alpha = \lambda \ominus \mu$. Because $i_k = -i_{k+1}$, it is easy to see that $\alpha \in \mathcal{AR}_\mu(i_{k+1})$. As $h_k(\mathbf{i}) = -1$, by (3.6), we have $\mathcal{AR}_\mu(i_{k+1}) = \{\alpha\}$. Hence, we have $\mathbf{t}(k+1) = \pm\alpha$, which implies $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$.

Because $h_k(\mathbf{i}) = -2$, by Lemma 3.10, there exists a unique $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$ and $\mathbf{s} \neq \mathbf{t}$. Let β and γ be positive nodes such that $\mathbf{s}(k) = \pm\beta$ and $\mathbf{t}(k-1) = \mathbf{s}(k-1) = \pm\gamma$. By the construction of up-down tableaux, we have $\alpha, \beta, \gamma \in \mathcal{AR}_\lambda(i_k)$.

Because $h_k(\mathbf{i}) = -2$, we have $\mathcal{AR}_\lambda(i_k) = 2$ by (3.5). Hence, as $\mathbf{s} \neq \mathbf{t}$, we have $\alpha \neq \beta$, which forces $\gamma = \alpha$ or $\gamma = \beta$. Therefore, we have either $\mathbf{s}(k-1) + \mathbf{s}(k) = 0$ or $\mathbf{t}(k-1) + \mathbf{t}(k) = 0$.

If $\mathbf{t}(k-1) + \mathbf{t}(k) \neq 0$, then we have $\mathbf{s}(k-1) + \mathbf{s}(k) = 0$. As $\mathbf{i} \in I_{k-1,0}^n \cap I_{k,0}^n$, we have

$$\begin{aligned} f_{\mathbf{t}}\epsilon_{k-1}^\mathcal{O}e(\mathbf{i})^\mathcal{O} &= 0, & f_{\mathbf{t}}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} &= (-1)^{a_k(\mathbf{i})}f_{\mathbf{t}} + P_k(\mathbf{t})^{-1}Q_k(\mathbf{s})^{-1}e_k(\mathbf{t}, \mathbf{s})f_{\mathbf{t}}, \\ f_{\mathbf{t}}\epsilon_{k-1}^\mathcal{O}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} &= 0, & f_{\mathbf{t}}\epsilon_k^\mathcal{O}\epsilon_{k-1}^\mathcal{O}e(\mathbf{i})^\mathcal{O} &= (-1)^{a_{k-1}(\mathbf{i})}P_k(\mathbf{t})^{-1}Q_k(\mathbf{s})^{-1}e_k(\mathbf{t}, \mathbf{s})f_{\mathbf{t}}, \end{aligned}$$

by Lemma 6.12, Lemma 7.2 and Lemma 7.4. Hence, $f_{\mathbf{t}}((-1)^{a_k(\mathbf{i})}\epsilon_k^\mathcal{O} + 2(-1)^{a_{k-1}(\mathbf{i})}\epsilon_{k-1}^\mathcal{O} - \epsilon_{k-1}^\mathcal{O}\epsilon_k^\mathcal{O} - \epsilon_k^\mathcal{O}\epsilon_{k-1}^\mathcal{O})e(\mathbf{i})^\mathcal{O}$ equals

$$f_{\mathbf{t}} + ((-1)^{a_k(\mathbf{i})} - (-1)^{a_{k-1}(\mathbf{i})})P_k(\mathbf{t})^{-1}Q_k(\mathbf{s})^{-1}e_k(\mathbf{t}, \mathbf{s})f_{\mathbf{t}}.$$

By Lemma 7.21, we have $(-1)^{a_k(\mathbf{i})} - (-1)^{a_{k-1}(\mathbf{i})} = 0$ because $\mathbf{i} \in I_{k,0}^n$. Hence, we have

$$f_{\mathbf{tt}}((-1)^{a_k(\mathbf{i})}\epsilon_k^{\mathcal{O}} + 2(-1)^{a_{k-1}(\mathbf{i})}\epsilon_{k-1} - \epsilon_{k-1}^{\mathcal{O}}\epsilon_k^{\mathcal{O}} - \epsilon_k^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}})e(\mathbf{i})^{\mathcal{O}} = f_{\mathbf{tt}}, \quad (7.11)$$

when $t(k-1) + t(k) \neq 0$.

If $t(k-1) + t(k) = 0$, then we have $s(k-1) + s(k) \neq 0$. As $\mathbf{i} \in I_{k-1,0}^n \cap I_{k,0}^n$, we have

$$\begin{aligned} f_{\mathbf{tt}}\epsilon_{k-1}^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} &= (-1)^{a_{k-1}(\mathbf{i})}f_{\mathbf{tt}}, & f_{\mathbf{tt}}\epsilon_k^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} &= (-1)^{a_k(\mathbf{i})}f_{\mathbf{tt}} + P_k(\mathbf{t})^{-1}Q_k(\mathbf{s})^{-1}e_k(\mathbf{t}, \mathbf{s})f_{\mathbf{ts}}, \\ f_{\mathbf{tt}}\epsilon_{k-1}^{\mathcal{O}}\epsilon_k^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} &= (-1)^{a_{k-1}(\mathbf{i})+a_k(\mathbf{i})}f_{\mathbf{tt}} + (-1)^{a_{k-1}(\mathbf{i})}P_k(\mathbf{t})^{-1}Q_k(\mathbf{s})^{-1}e_k(\mathbf{t}, \mathbf{s})f_{\mathbf{ts}}, & f_{\mathbf{tt}}\epsilon_k^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} &= (-1)^{a_{k-1}(\mathbf{i})+a_k(\mathbf{i})}f_{\mathbf{tt}}, \end{aligned}$$

by Lemma 6.12, Lemma 7.2 and Lemma 7.4. Hence, $f_{\mathbf{tt}}((-1)^{a_k(\mathbf{i})}\epsilon_k^{\mathcal{O}} + 2(-1)^{a_{k-1}(\mathbf{i})}\epsilon_{k-1} - \epsilon_{k-1}^{\mathcal{O}}\epsilon_k^{\mathcal{O}} - \epsilon_k^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}})e(\mathbf{i})^{\mathcal{O}}$ equals

$$((-1)^{2a_k(\mathbf{i})} + 2(-1)^{2a_{k-1}(\mathbf{i})} - 2(-1)^{a_{k-1}(\mathbf{i})+a_k(\mathbf{i})})f_{\mathbf{tt}} + ((-1)^{a_k(\mathbf{i})} - (-1)^{a_{k-1}(\mathbf{i})})P_k(\mathbf{t})^{-1}Q_k(\mathbf{s})^{-1}e_k(\mathbf{t}, \mathbf{s})f_{\mathbf{ts}}.$$

By Lemma 7.21, we have $(-1)^{a_k(\mathbf{i})} = (-1)^{a_{k-1}(\mathbf{i})}$ because $\mathbf{i} \in I_{k,0}^n$. Hence, we have

$$f_{\mathbf{tt}}((-1)^{a_k(\mathbf{i})}\epsilon_k^{\mathcal{O}} + 2(-1)^{a_{k-1}(\mathbf{i})}\epsilon_{k-1} - \epsilon_{k-1}^{\mathcal{O}}\epsilon_k^{\mathcal{O}} - \epsilon_k^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}})e(\mathbf{i})^{\mathcal{O}} = f_{\mathbf{tt}}, \quad (7.12)$$

when $t(k-1) + t(k) = 0$.

By (7.11) and (7.12), we have

$$f_{\mathbf{tt}}((-1)^{a_k(\mathbf{i})}\epsilon_k^{\mathcal{O}} + 2(-1)^{a_{k-1}(\mathbf{i})}\epsilon_{k-1} - \epsilon_{k-1}^{\mathcal{O}}\epsilon_k^{\mathcal{O}} - \epsilon_k^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}})e(\mathbf{i})^{\mathcal{O}} = f_{\mathbf{tt}},$$

which implies

$$e(\mathbf{i})^{\mathcal{O}} = (-1)^{a_k(\mathbf{i})}e(\mathbf{i})^{\mathcal{O}}\epsilon_k^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} + 2(-1)^{a_{k-1}(\mathbf{i})}e(\mathbf{i})^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} - e(\mathbf{i})^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}}\epsilon_k^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} - e(\mathbf{i})^{\mathcal{O}}\epsilon_k^{\mathcal{O}}\epsilon_{k-1}^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}} \in \mathcal{B}_n^{\mathcal{O}}(x).$$

by Lemma 6.1. Hence the Lemma holds by lifting the elements into $\mathcal{B}_n(\delta)$. \square

7.23. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in I_{k,-}^n$ with $i_k + i_{k+1} = 0$. Then we have $e(\mathbf{i})(\epsilon_k y_k + y_k \epsilon_k)e(\mathbf{i}) = (-1)^{a_k(\mathbf{i})}e(\mathbf{i})$.

Proof. Choose an arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and write $\lambda = \mathbf{t}_k$, $\mu = \mathbf{t}_{k+1}$. As $\mathbf{i} \in I_{k,-}^n$, we have $h_k(\mathbf{i}) = -2$, and $h_{k+1}(\mathbf{i}) = -1$ if $i_k = \pm \frac{1}{2}$ and $h_{k+1}(\mathbf{i}) = 0$ if $i_k \neq \pm \frac{1}{2}$. In both cases, by (3.6) and (3.7), we have $|\mathcal{A}\mathcal{R}_{\mu}(i_{k+1})| = 1$. Let $\alpha = \lambda \ominus \mu$. Because $i_k + i_{k+1} = 0$, by the construction of up-down tableaux, we have $\alpha \in \mathcal{A}\mathcal{R}_{\mu}(i_{k+1})$. Hence, it forces $t(k) + t(k+1) = 0$.

Because $h_k(\mathbf{i}) = -2$, by Lemma 3.10, there exists a unique $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{t}$ and $\mathbf{s} \neq \mathbf{t}$, and $c_t(k) - i_k = -(c_s(k) - i_k)$. Hence, by Lemma 6.12, we have

$$\begin{aligned} f_{\mathbf{tt}}(\epsilon_k^{\mathcal{O}} y_k^{\mathcal{O}} + y_k^{\mathcal{O}} \epsilon_k^{\mathcal{O}})e(\mathbf{i})^{\mathcal{O}} &= 2(c_t(k) - i_k)f_{\mathbf{tt}}\epsilon_k^{\mathcal{O}}\frac{f_{\mathbf{tt}}}{\gamma_t} + (c_s(k) - i_k + c_t(k) - i_k)f_{\mathbf{tt}}\epsilon_k^{\mathcal{O}}\frac{f_{\mathbf{ss}}}{\gamma_s} \\ &= 2(c_t(k) - i_k)f_{\mathbf{tt}}\epsilon_k^{\mathcal{O}}\frac{f_{\mathbf{tt}}}{\gamma_t} = (-1)^{a_k(\mathbf{i})}f_{\mathbf{tt}}. \end{aligned}$$

As \mathbf{t} is chosen arbitrary, by Lemma 6.1, we have

$$e(\mathbf{i})^{\mathcal{O}}(\epsilon_k^{\mathcal{O}} y_k^{\mathcal{O}} + y_k^{\mathcal{O}} \epsilon_k^{\mathcal{O}})e(\mathbf{i})^{\mathcal{O}} = (-1)^{a_k(\mathbf{i})}e(\mathbf{i})^{\mathcal{O}} \in \mathcal{B}_n^{\mathcal{O}}(x),$$

which proves the Lemma by lifting the elements into $\mathcal{B}_n(\delta)$. \square

Next we prove the relation (3.20) holds in $\mathcal{B}_n(\delta)$. We remind reader that when we prove this relation we do not assume $\mathbf{i} \in I^n$. In this relation, when $\mathbf{i} \notin I^n$, we have $e(\mathbf{j})\epsilon_k e(\mathbf{i})\epsilon_k e(\mathbf{k}) = 0$ because $e(\mathbf{i}) = 0$ by Lemma 6.1, but it is not easy to see the left hand side of the relation equals 0, except when $\mathbf{i} \in P_{k,-}^n$. Here we start by proving the case when $\mathbf{i} \in P_{k,-}^n$.

7.24. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in P_{k,-}^n$, $\mathbf{j}, \mathbf{k} \in I^n$. Then we have $e(\mathbf{j})\epsilon_k e(\mathbf{i})\epsilon_k e(\mathbf{k}) = 0$.

Proof. When $\mathbf{i} \notin I^n$, we have $e(\mathbf{i}) = 0$ by Lemma 6.1, which implies $e(\mathbf{j})\epsilon_k e(\mathbf{i})\epsilon_k e(\mathbf{k}) = 0$. Hence we assume $\mathbf{i} \in I^n$ in the rest of the proof. Note that $\mathbf{i} \in I^n$ and $\mathbf{i} \in P_{k,-}^n$ is equivalent to $\mathbf{i} \in I_{k,-}^n$.

Choose arbitrary $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j})$. If $u(k) + u(k+1) \neq 0$, we have

$$f_{\mathbf{uu}}\epsilon_k^{\mathcal{O}}e(\mathbf{i})^{\mathcal{O}}\epsilon_k^{\mathcal{O}}e(\mathbf{k})^{\mathcal{O}} = 0. \quad (7.13)$$

Assume $u(k) + u(k+1) = 0$. Because $\mathbf{i} \in I_{k,-}^n$, we have $h_k(\mathbf{i}) = -2$. By Lemma 3.10, there exist exactly two up-down tableaux $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{u} \stackrel{k}{\sim} \mathbf{t}$, and $c_t(k) - i_k = -(c_s(k) - i_k)$. Therefore,

by Theorem 2.18, Lemma 7.4 and Lemma 6.12, we have

$$\begin{aligned} f_{uu} \epsilon_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{k})^{\mathcal{O}} &= \sum_{\substack{\mathbf{v} \in \mathcal{T}_n^{ud}(\mathbf{k}) \\ \mathbf{v} \stackrel{k}{\sim} \mathbf{t}}} P_k(\mathbf{u})^{-1} Q_k(\mathbf{v})^{-1} e_k(\mathbf{u}, \mathbf{v}) (P_k(\mathbf{t})^{-1} Q_k(\mathbf{t})^{-1} e_k(\mathbf{t}, \mathbf{t}) + P_k(\mathbf{s})^{-1} Q_k(\mathbf{s})^{-1} e_k(\mathbf{s}, \mathbf{s})) f_{uv} \\ &= \sum_{\substack{\mathbf{v} \in \mathcal{T}_n^{ud}(\mathbf{k}) \\ \mathbf{v} \stackrel{k}{\sim} \mathbf{t}}} P_k(\mathbf{u})^{-1} Q_k(\mathbf{v})^{-1} e_k(\mathbf{u}, \mathbf{v}) \left(\frac{1}{2(c_t(k) - i_k)} + \frac{1}{2(c_s(k) - i_k)} \right) f_{uv} = 0. \end{aligned}$$

By combining the above equality and (7.13), we have $f_{uu} \epsilon_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{k})^{\mathcal{O}} = 0$, and by Lemma 6.1, we have $e(\mathbf{j})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{k})^{\mathcal{O}} = 0$. The Lemma follows by lifting the elements into $\mathcal{B}_n(\delta)$. \square

Next we prove the cases when $\mathbf{i} \in P_{k,0}^n$. Recall

$$z_k(\mathbf{i}) = \begin{cases} 0, & \text{if } h_k(\mathbf{i}) < -2, \text{ or } h_k(\mathbf{i}) \geq 0 \text{ and } i_k \neq 0, \\ (-1)^{a_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}), & \text{if } -2 \leq h_k(\mathbf{i}) < 0, \\ \frac{1 + (-1)^{a_k(\mathbf{i})}}{2}, & \text{if } i_k = 0. \end{cases}$$

The key point of $z_k(\mathbf{i})$ is given in the next Lemma.

7.25. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in P_{k,0}^n$, $\mathbf{j} \in I^n$ with $\mathbf{i}|_{k-1} = \mathbf{j}|_{k-1}$. For any $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j})$ with $u(k) + u(k+1) = 0$, we have the following properties:

- (1) If $h_k(\mathbf{i}) < -2$, or $h_k(\mathbf{i}) \geq 0$ and $i_k \neq 0$, then there does not exist $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$.
- (2) If $-2 \leq h_k(\mathbf{i}) < 0$, then there exists $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$.
- (3) If $i_k = 0$, then

$$z_k(\mathbf{i}) = \begin{cases} 0, & \text{if there does not exist } \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i}) \text{ such that } \mathbf{t} \stackrel{k}{\sim} \mathbf{u}, \\ (-1)^{a_k(\mathbf{i})}, & \text{if there exists } \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i}) \text{ such that } \mathbf{t} \stackrel{k}{\sim} \mathbf{u}. \end{cases}$$

Proof. Write $\lambda = \mathbf{u}_{k-1}$.

(1). By Lemma 3.6, we have $\mathbf{i} \notin I^n$ if $h_k(\mathbf{i}) < -2$ or $h_k(\mathbf{i}) > 0$. When $h_k(\mathbf{i}) = 0$, assume $i_k \neq -\frac{1}{2}$. Then we have $\mathbf{i} \in P_{k,+}^n$ by the definition and $i_k \neq 0$, which contradicts to the assumptions of the Lemma. Hence we have $i_k = -\frac{1}{2}$, which implies $\mathbf{i} \notin I^n$ by Lemma 3.7. Therefore, in this case we always have $\mathbf{i} \notin I^n$, which implies there does not exist $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$.

(2). In this case we have $h_k(\mathbf{i}) = -1$ or -2 . By Lemma 3.4 and (3.4) it forces $|\mathcal{AR}_\lambda(i_k)| \geq 1$. Let $\alpha \in \mathcal{AR}_\lambda(i_k)$. Without loss of generality we assume $\alpha \in \mathcal{A}(\lambda)$. Hence let \mathbf{t} be the up-down tableau such that $\mathbf{t}(k) = \alpha$, $\mathbf{t}(k+1) = -\alpha$ and $\mathbf{t}(\ell) = \mathbf{u}(\ell)$ for any $\ell \neq k, k+1$. Therefore $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$.

(3). Because $i_k = 0$, we have $h_k(\mathbf{i}) = 0$ by the definition of h_k . By Lemma 3.4 and (3.4) it forces $|\mathcal{AR}_\lambda(i_k)| = 0$ or 1. If there exists $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$, we require $|\mathcal{AR}_\lambda(0)| = 1$, which implies $z_k(\mathbf{i}) = 1 = (-1)^{a_k(\mathbf{i})}$ by Corollary 3.16; and if there does not exist $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$, we have $|\mathcal{AR}_\lambda(0)| = 0$, which implies $z_k(\mathbf{i}) = 0$ by Corollary 3.16. \square

Therefore, under the assumptions of Lemma 7.25 we can rewrite $z_k(\mathbf{i})$ as

$$z_k(\mathbf{i}) = \begin{cases} 0, & \text{if there does not exist } \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i}) \text{ such that } \mathbf{t} \stackrel{k}{\sim} \mathbf{u}, \\ (-1)^{a_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}), & \text{if there exists } \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i}) \text{ such that } \mathbf{t} \stackrel{k}{\sim} \mathbf{u}, \end{cases} \quad (7.14)$$

which allows us to verify relation (3.20) when $\mathbf{i} \in P_{k,0}^n$ via direct calculations.

7.26. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in P_{k,0}^n$, $\mathbf{j}, \mathbf{k} \in I^n$. For any $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j})$ with $u(k) + u(k+1) = 0$, we have

$$f_{uu} \epsilon_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{k})^{\mathcal{O}} = z_k(\mathbf{i}) f_{uu} \epsilon_k^{\mathcal{O}} e(\mathbf{k})^{\mathcal{O}}.$$

Proof. Suppose there does not exist $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$. Then by Lemma 7.2 and (7.14) we have

$$f_{uu} \epsilon_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{k})^{\mathcal{O}} = 0 = z_k(\mathbf{i}) f_{uu} \epsilon_k^{\mathcal{O}} e(\mathbf{k})^{\mathcal{O}}.$$

Suppose there exists $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$. It requires $\mathbf{i} \in I_{k,0}^n$. When $h_k(\mathbf{i}) = -1$, we have $i_k \neq -\frac{1}{2}$, otherwise $\mathbf{i} \in P_{k,+}^n$ which contradicts to the assumptions of the Lemma. Therefore we have $z_k(\mathbf{i}) = (-1)^{a_k(\mathbf{i})}$.

By Lemma 3.10, there exists a unique $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$. Hence, by Theorem 2.18, Lemma 7.4 and Lemma 6.12, we have

$$\begin{aligned} f_{uu} \epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta &= \sum_{\substack{\mathbf{v} \in \mathcal{T}_n^{ud}(\mathbf{k}) \\ \mathbf{v} \stackrel{k}{\sim} \mathbf{t}}} P_k(\mathbf{u})^{-1} e_k(\mathbf{u}, \mathbf{t}) Q_k(\mathbf{t})^{-1} P_k(\mathbf{t})^{-1} Q_k(\mathbf{v})^{-1} e_k(\mathbf{t}, \mathbf{v}) f_{uv} \\ &= (-1)^{a_k(\mathbf{i})} \sum_{\substack{\mathbf{v} \in \mathcal{T}_n^{ud}(\mathbf{k}) \\ \mathbf{v} \stackrel{k}{\sim} \mathbf{u}}} P_k(\mathbf{u})^{-1} e_k(\mathbf{u}, \mathbf{v}) Q_k(\mathbf{v})^{-1} f_{uv} = (-1)^{a_k(\mathbf{i})} f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta = z_k(\mathbf{i}) f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta. \end{aligned}$$

When $h_k(\mathbf{i}) = 0$, we have $i_k = 0$. Hence following the same argument as when $h_k(\mathbf{i}) = -1$, we have

$$f_{uu} \epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = (-1)^{a_k(\mathbf{i})} f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta = z_k(\mathbf{i}) f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta.$$

When $h_k(\mathbf{i}) = -2$, we have $i_k = -\frac{1}{2}$. Hence following the same argument as when $h_k(\mathbf{i}) = -1$, we have

$$f_{uu} \epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = 2(-1)^{a_k(\mathbf{i})} f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta = z_k(\mathbf{i}) f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta,$$

where the coefficient is doubled because we have $\mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{s} \stackrel{k}{\sim} \mathbf{u} \stackrel{k}{\sim} \mathbf{t}$, and when $h_k(\mathbf{i}) = -1$, there is a unique $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$. \square

7.27. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i} \in P_{k,0}^n$, $\mathbf{j}, \mathbf{k} \in I^n$. Then we have $e(\mathbf{j}) \epsilon_k e(\mathbf{i}) \epsilon_k e(\mathbf{k}) = z_k(\mathbf{i}) e(\mathbf{j}) \epsilon_k e(\mathbf{k})$.

Proof. Choose arbitrary $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j})$. If $u(k) + u(k+1) \neq 0$, we have $f_{uu} \epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = 0 = z_k(\mathbf{i}) f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta$; and if $u(k) + u(k+1) = 0$, we have $f_{uu} \epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = z_k(\mathbf{i}) f_{uu} \epsilon_k^\theta e(\mathbf{k})^\theta$ by Lemma 7.26. Therefore, we have

$$e(\mathbf{j}) \epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = z_k(\mathbf{i}) e(\mathbf{j}) \epsilon_k^\theta e(\mathbf{k})^\theta \in \mathcal{B}_n^\theta(x),$$

by Lemma 6.1, which proves the Lemma by lifting the elements into $\mathcal{B}_n(\delta)$. \square

To prove (3.20) when $\mathbf{i} \in P_{k,+}^n$, we want to give a result analogue to Lemma 7.25. In more details, we want to prove the next Lemma:

7.28. Lemma. Suppose $\mathbf{i} \in P_{k,+}^n$ and $\mathbf{j} \in I^n$ with $\mathbf{i} \stackrel{k}{\sim} \mathbf{j}$ for some $1 \leq k \leq n$. For any $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j})$ with $u(k) + u(k+1) = 0$, we have

$$\begin{aligned} &(1 + \delta_{i_k, -\frac{1}{2}}) \left(\sum_{\ell \in A_{k,1}^{\mathbf{i}}} y_\ell^\theta - 2 \sum_{\ell \in A_{k,2}^{\mathbf{i}}} y_\ell^\theta + \sum_{\ell \in A_{k,3}^{\mathbf{i}}} y_\ell^\theta - 2 \sum_{\ell \in A_{k,4}^{\mathbf{i}}} y_\ell^\theta \right) f_{uu} \\ &= \begin{cases} 0, & \text{if there does not exist } \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i}) \text{ such that } \mathbf{t} \stackrel{k}{\sim} \mathbf{u}, \\ 2(c_t(k) - i_k) f_{uu}, & \text{if there exists } \mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i}) \text{ such that } \mathbf{t} \stackrel{k}{\sim} \mathbf{u}. \end{cases} \end{aligned}$$

Under the assumptions of Lemma 7.28, write $\lambda = u_{k-1}$. Lemma 7.28 explicitly express the connection between λ and $\mathcal{AR}_\lambda(i_k)$. Hence we express such connection before we prove Lemma 7.28.

For $1 \leq k \leq n$, fix $\mathbf{i} \in P_{k,+}^n$, $\mathbf{j} \in I^n$ with $\mathbf{i} \stackrel{k}{\sim} \mathbf{j}$ and $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j})$. For convenience we write $i = i_k \in P$. Because $\mathbf{i} \in P_{k,+}^n$ we have $-1 \leq h_k(\mathbf{i}) \leq 0$.

Let $u_{k-1} = \lambda$. By Lemma 3.4 and (3.4), we have three possible cases:

- (1) $h_k(\mathbf{i}) = 0$ and $|\mathcal{AR}_\lambda(-i)| = |\mathcal{AR}_\lambda(i)| = 0$.
- (2) $h_k(\mathbf{i}) = 0$ and $|\mathcal{AR}_\lambda(-i)| = |\mathcal{AR}_\lambda(i)| = 1$.
- (3) $h_k(\mathbf{i}) = -1$ and $|\mathcal{AR}_\lambda(-i)| = 0$, $|\mathcal{AR}_\lambda(i)| = 1$.

For $\alpha \in \mathcal{AR}(\lambda)$, define

$$\text{cont}_\lambda(\alpha) = \begin{cases} \text{cont}(\alpha), & \text{if } \alpha \in \mathcal{A}(\lambda), \\ -\text{cont}(\alpha), & \text{if } \alpha \in \mathcal{R}(\lambda), \end{cases}$$

and denote $\text{cont}_\lambda(i) = \text{cont}_\lambda(\alpha)$ where $\alpha \in \mathcal{AR}_\lambda(i)$ if $|\mathcal{AR}_\lambda(i)| = 1$, and $\text{cont}_\lambda(i) = \frac{x-\delta}{2} + i$ if $|\mathcal{AR}_\lambda(i)| = 0$.

By Lemma 3.3 and Lemma 2.17, set u as an unknown and we have

$$\frac{u + c_u(1)}{u - c_u(1)} \prod_{\ell=1}^{k-1} \frac{u + c_u(\ell) + 1}{u - c_u(\ell) + 1} \frac{u + c_u(\ell) - 1}{u - c_u(\ell) - 1} \frac{(u - c_u(\ell))^2}{(u + c_u(\ell))^2} = \prod_{\alpha \in \mathcal{AR}(\lambda)} \frac{u + \text{cont}_\lambda(\alpha)}{u - \text{cont}_\lambda(\alpha)}. \quad (7.15)$$

We note that the right hand side of (7.15) contains information of $\mathcal{AR}_\lambda(i)$, which is accessible from the left hand side of (7.15).

Define subsets of $R[u, x]$ as

$$\begin{aligned} S_1(u) &:= \{u + c_u(1)\} \cup \bigcup_{\ell=1}^{k-1} \{u + c_u(\ell) + 1, u + c_u(\ell) - 1\}, & S_2(u) &:= \{u - c_u(\ell) \mid 1 \leq \ell \leq k-1\}, \\ S_3(u) &:= \{u - c_u(1)\} \cup \bigcup_{\ell=1}^{k-1} \{u - c_u(\ell) + 1, u - c_u(\ell) - 1\}, & S_4(u) &:= \{u + c_u(\ell) \mid 1 \leq \ell \leq k-1\}, \\ T_1(u) &:= \{u + \text{cont}_\lambda(\alpha) \mid \alpha \in \mathcal{AR}(\lambda)\}, & T_2(u) &:= \{u - \text{cont}_\lambda(\alpha) \mid \alpha \in \mathcal{AR}(\lambda)\}, \end{aligned}$$

By the definitions, for $w[u, x] \in \bigcup_{r=1}^4 S_{k,r}$, we have $w(i, \delta) \in R$. Define

$$\begin{aligned} S_1^i(u) &:= \{w(u, x) \in S_1(u) \mid w(i, \delta) = 0\}, & S_2^i(u) &:= \{w(u, x) \in S_2(u) \mid w(i, \delta) = 0\}, \\ S_3^i(u) &:= \{w(u, x) \in S_3(u) \mid w(i, \delta) = 0\}, & S_4^i(u) &:= \{w(u, x) \in S_4(u) \mid w(i, \delta) = 0\}, \\ T_1^i(u) &:= \{w(u, x) \in T_1(u) \mid w(i, \delta) = 0\}, & T_2^i(u) &:= \{w(u, x) \in T_2(u) \mid w(i, \delta) = 0\}. \end{aligned}$$

7.29. Remark. By the definitions, we have $c_u(\ell) = \pm \frac{x-\delta}{2} + j$ for some $j \in P$. Therefore, for $w(u, x) \in \bigcup_{r=1}^4 S_r(u)$, we have $w(u, x) = u \pm \frac{x-\delta}{2} + j$ for some $j \in P$. Hence, $w(i, \delta) = 0$ only if $w(u, x) = u \pm \frac{x-\delta}{2} - i$.

By the definitions, we have $\text{cont}_\lambda(i) = \pm \frac{x-\delta}{2} + i$. Therefore, for $w(u, x) \in \bigcup_{r=1}^4 S_r^i(u)$, we have $w(\text{cont}_\lambda(i), x) = 2(\text{cont}_\lambda(i) - i)$ or 0. Similarly, we have the same property for $w(u, x) \in T_1^i(u) \cup T_2^i(u)$.

7.30. Lemma. We have the following equality

$$\frac{\prod_{w \in S_1^i(u)} w \prod_{w \in S_2^i(u)} w^2}{\prod_{w \in S_3^i(u)} w \prod_{w \in S_4^i(u)} w^2} = \frac{\prod_{w \in T_1^i(u)} w}{\prod_{w \in T_2^i(u)} w}.$$

Proof. For convenience, we define

$$\begin{aligned} f_1(u, x) &= \frac{\prod_{w \in S_1(u)} w \prod_{w \in S_2(u)} w^2}{\prod_{w \in S_3(u)} w \prod_{w \in S_4(u)} w^2}, & f_2(u, x) &= \frac{\prod_{w \in T_1(u)} w}{\prod_{w \in T_2(u)} w}, \\ g_1(u, x) &= \frac{\prod_{w \in S_1^i(u)} w \prod_{w \in S_2^i(u)} w^2}{\prod_{w \in S_3^i(u)} w \prod_{w \in S_4^i(u)} w^2}, & g_2(u, x) &= \frac{\prod_{w \in T_1^i(u)} w}{\prod_{w \in T_2^i(u)} w}, \end{aligned}$$

and $u_1(u, x) = f_1(u, x)/g_1(u, x)$ and $u_2(u, x) = f_2(u, x)/g_2(u, x)$.

By (7.15), we have $f_1(u, x) = f_2(u, x)$. Hence, by Remark 7.29, we can write

$$f_1(u, \delta) = f_2(u, \delta) = \prod_{j \in P} (u - j)^{a_j},$$

where $a_j \in \mathbb{Z}$. By the definitions, we have $u_1(i, \delta) \neq 0$ and $u_2(i, \delta) \neq 0$. Therefore, we have

$$u_1(u, \delta) = \prod_{\substack{j \in P \\ j \neq i}} (u - j)^{a_j} = u_2(u, \delta),$$

which yields $g_1(u, \delta) = g_2(u, \delta) = (u - i)^{a_i}$.

By Remark 7.29, we can write

$$g_1(u, x) = (u + \frac{\delta-1}{2} - i)^{k_1} (u - \frac{\delta-1}{2} - i)^{l_1} \quad \text{and} \quad g_2(u, x) = (u + \frac{\delta-1}{2} - i)^{k_2} (u - \frac{\delta-1}{2} - i)^{l_2},$$

where $k_1, l_1, k_2, l_2 \in \mathbb{Z}$ and $k_1 + l_1 = k_2 + l_2 = a_i$. If we have $k_1 > k_2$, then by defining

$$s_1(u, x) = (u + \frac{\delta-1}{2} - i)^{-k_2} f_1(u, x), \quad \text{and} \quad s_2(u, x) = (u + \frac{\delta-1}{2} - i)^{-k_2} f_2(u, x),$$

we have $s_1(-\frac{\delta-1}{2} + i, x) = 0$ and $s_2(-\frac{\delta-1}{2} + i, x) \neq 0$, which contradicts the fact that $f_1(u, x) = f_2(u, x)$. Hence, we have $k_1 \leq k_2$. Similarly, we have $k_1 \geq k_2$, which implies $k_1 = k_2$. Therefore, we have $l_1 = l_2$, and $g_1(u, x) = g_2(u, x)$. \square

7.31. Lemma. We have the following equality

$$\frac{\prod_{w \in S_1^i(u)} w \prod_{w \in S_2^i(u)} w^2}{\prod_{w \in S_3^i(u)} w \prod_{w \in S_4^i(u)} w^2} = \begin{cases} \frac{1}{u - \text{cont}_\lambda(i)}, & \text{if } h_k(\mathbf{i}) = -1, \\ \frac{u + \text{cont}_\lambda(i) - 2i}{u - \text{cont}_\lambda(i)}, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{AR}_\lambda(i)| = 1, \\ 1, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{AR}_\lambda(i)| = 0. \end{cases}$$

Proof. By Lemma 7.30, it suffices to prove that

$$\frac{\prod_{w \in T_1^i(u)} w}{\prod_{w \in T_2^i(u)} w} = \begin{cases} \frac{1}{u - \text{cont}_\lambda(i)}, & \text{if } h_k(\mathbf{i}) = -1, \\ \frac{u + \text{cont}_\lambda(i) - 2i}{u - \text{cont}_\lambda(i)}, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 1, \\ 1, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 0. \end{cases}$$

For $h_k(\mathbf{i}) = -1$, by (3.6), we have $|\mathcal{A}\mathcal{R}_\lambda(i)| = 1$. Hence, there exists a unique $\alpha \in \mathcal{A}\mathcal{R}_\lambda(i)$. By the definition, we have $\text{cont}_\lambda(i) = \text{cont}_\lambda(\alpha)$. Hence, we have

$$\frac{\prod_{w \in T_1^i(u)} w}{\prod_{w \in T_2^i(u)} w} = \frac{1}{u - \text{cont}_\lambda(\alpha)} = \frac{1}{u - \text{cont}_\lambda(i)}.$$

For $h_k(\mathbf{i}) = 0$ and $|\mathcal{A}\mathcal{R}_\lambda(i)| = 1$, by (3.7), we have $|\mathcal{A}\mathcal{R}_\lambda(i)| = |\mathcal{A}\mathcal{R}_\lambda(-i)| = 1$. Therefore, there exist $\alpha \in \mathcal{A}\mathcal{R}_\lambda(i)$ and $\beta \in \mathcal{A}\mathcal{R}_\lambda(-i)$. By the definition, we have $\text{cont}_\lambda(i) = \text{cont}_\lambda(\alpha)$. If $i = 0$, we have $\alpha = \beta$ because $i = -i$. Hence, we have

$$\frac{\prod_{w \in T_1^i(u)} w}{\prod_{w \in T_2^i(u)} w} = \frac{u + \text{cont}_\lambda(\alpha)}{u - \text{cont}_\lambda(\alpha)} = \frac{u + \text{cont}_\lambda(i) - 2i}{u - \text{cont}_\lambda(i)},$$

and if $i \neq 0$, by Lemma 3.8, we have $\alpha, \beta \in \mathcal{A}(\lambda)$ or $\alpha, \beta \in \mathcal{R}(\lambda)$. Therefore, we have $\text{cont}_\lambda(\alpha) - i = \text{cont}_\lambda(\beta) + i$, which implies $\text{cont}_\lambda(\beta) = \text{cont}_\lambda(\alpha) - 2i = \text{cont}_\lambda(i) - 2i$. Hence, we have

$$\frac{\prod_{w \in T_1^i(u)} w}{\prod_{w \in T_2^i(u)} w} = \frac{u + \text{cont}_\lambda(\beta)}{u - \text{cont}_\lambda(\alpha)} = \frac{u + \text{cont}_\lambda(i) - 2i}{u - \text{cont}_\lambda(i)}.$$

For $h_k(\mathbf{i}) = 0$ and $|\mathcal{A}\mathcal{R}_\lambda(i)| = 0$, we have $|\mathcal{A}\mathcal{R}_\lambda(i)| = |\mathcal{A}\mathcal{R}_\lambda(-i)| = 0$. Hence, we have

$$\frac{\prod_{w \in T_1^i(u)} w}{\prod_{w \in T_2^i(u)} w} = 1,$$

which completes the proof. \square

Lemma 7.31 gives us a method to determine $|\mathcal{A}\mathcal{R}_\lambda(i)|$ by giving λ , but it is not sufficient to determine whether $\alpha \in \mathcal{A}(\lambda)$ or $\alpha \in \mathcal{R}(\lambda)$ for $\alpha \in \mathcal{A}\mathcal{R}_\lambda(i)$ given $|\mathcal{A}\mathcal{R}_\lambda(i)| = 1$.

Recall that by Remark 7.29, for $w(u, x) \in \bigcup_{r=1}^4 S_r^i(u)$, we have $w(\text{cont}_\lambda(i), x) = 2(\text{cont}_\lambda(i) - i)$ or 0. Define

$$\begin{aligned} a_{1,1} &= \# \{ w \in S_1^i(u) \mid w(\text{cont}_\lambda(i), x) = 2(\text{cont}_\lambda(i) - i) \}, & a_{1,2} &= \# \{ w \in S_1^i(u) \mid w(\text{cont}_\lambda(i), x) = 0 \}, \\ a_{2,1} &= \# \{ w \in S_2^i(u) \mid w(\text{cont}_\lambda(i), x) = 2(\text{cont}_\lambda(i) - i) \}, & a_{2,2} &= \# \{ w \in S_2^i(u) \mid w(\text{cont}_\lambda(i), x) = 0 \}, \\ a_{3,1} &= \# \{ w \in S_3^i(u) \mid w(\text{cont}_\lambda(i), x) = 2(\text{cont}_\lambda(i) - i) \}, & a_{3,2} &= \# \{ w \in S_3^i(u) \mid w(\text{cont}_\lambda(i), x) = 0 \}, \\ a_{4,1} &= \# \{ w \in S_4^i(u) \mid w(\text{cont}_\lambda(i), x) = 2(\text{cont}_\lambda(i) - i) \}, & a_{4,2} &= \# \{ w \in S_4^i(u) \mid w(\text{cont}_\lambda(i), x) = 0 \}. \end{aligned}$$

By Lemma 7.31, we have

$$\begin{aligned} a_{1,1} + 2a_{2,1} - a_{3,1} - 2a_{4,1} &= \begin{cases} 1, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 1, \\ 0, & \text{otherwise;} \end{cases} \\ a_{1,2} + 2a_{2,2} - a_{3,2} - 2a_{4,2} &= \begin{cases} 0, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 0, \\ -1, & \text{otherwise.} \end{cases} \end{aligned} \tag{7.16}$$

7.32. Lemma. *We have the following equalities:*

$$\begin{aligned} \sum_{\ell \in A_{k,1}^{\mathbf{i}}} y_\ell^\theta f_{uu} &= (a_{1,1}(\text{cont}_\lambda(i) - i) - a_{1,2}(\text{cont}_\lambda(i) - i)) f_{uu}, \\ \sum_{\ell \in A_{k,2}^{\mathbf{i}}} y_\ell^\theta f_{uu} &= (a_{2,2}(\text{cont}_\lambda(i) - i) - a_{2,1}(\text{cont}_\lambda(i) - i)) f_{uu}, \\ \sum_{\ell \in A_{k,3}^{\mathbf{i}}} y_\ell^\theta f_{uu} &= (a_{3,2}(\text{cont}_\lambda(i) - i) - a_{3,1}(\text{cont}_\lambda(i) - i)) f_{uu}, \\ \sum_{\ell \in A_{k,4}^{\mathbf{i}}} y_\ell^\theta f_{uu} &= (a_{4,1}(\text{cont}_\lambda(i) - i) - a_{4,2}(\text{cont}_\lambda(i) - i)) f_{uu}. \end{aligned}$$

Proof. We will only prove the first equality. The rest equalities can be proved following the same argument.

For any polynomial $p(y_1^\theta, \dots, y_{k-1}^\theta)$, we write $p(y_1^\theta, \dots, y_{k-1}^\theta) f_{uu} = \alpha_u f_{uu}$. Recall

$$A_{k,1}^{\mathbf{i}} = \{ 1 \leq \ell \leq k-1 \mid i_\ell = -i-1 \text{ or } -i+1, \text{ or } \ell = 1 \text{ and } i_\ell = -i \}.$$

Hence $\sum_{\ell \in A_{k,1}^i} y_\ell^\theta$ is a polynomial of $y_1^\theta, \dots, y_{k-1}^\theta$. Therefore, it suffices to prove that $\alpha_u = a_{1,1}(\text{cont}_\lambda(i) - i) - a_{1,2}(\text{cont}_\lambda(i) - i)$.

Choose any $\ell \in A_{k,1}^i$. If $\ell = 1$, there is a unique $w_\ell \in \{u + c_u(1), u + c_u(1) + 1, u + c_u(1) - 1\}$ such that $w_\ell \in S_1^i(u)$. Similarly, if $\ell > 1$, there is a unique $w_\ell \in \{u + c_u(\ell) + 1, u + c_u(\ell) - 1\}$ such that $w_\ell \in S_1^i(u)$. Hence, for any $1 \leq \ell \leq k - 1$, we have

$$y_\ell^\theta f_{uu} = \begin{cases} (\text{cont}_\lambda(i) - i)f_{uu}, & \text{if } w_\ell(\text{cont}_\lambda(i), x) = 2(\text{cont}_\lambda(i) - i), \\ -(\text{cont}_\lambda(i) - i)f_{uu}, & \text{if } w_\ell(\text{cont}_\lambda(i), x) = 0. \end{cases}$$

Hence by the definitions of $a_{1,1}$ and $a_{1,2}$, we have $\alpha_u = a_{1,1}(\text{cont}_\lambda(i) - i) - a_{1,2}(\text{cont}_\lambda(i) - i)$. \square

Now we are ready to prove Lemma 7.28.

Proof of Lemma 7.28. By Lemma 7.32, we have

$$\begin{aligned} & \left(\sum_{\ell \in A_{k,1}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,2}^i} y_\ell^\theta + \sum_{\ell \in A_{k,3}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,4}^i} y_\ell^\theta \right) f_{tt} \\ &= (a_{1,1} - a_{1,2} - 2a_{2,2} + 2a_{2,1} + a_{3,2} - a_{3,1} - 2a_{4,1} + 2a_{4,2})(\text{cont}_\lambda(i) - i)f_{tt} \\ &= ((a_{1,1} + 2a_{2,1} - a_{3,1} - 2a_{4,1}) - (a_{1,2} + 2a_{2,2} - a_{3,2} - 2a_{4,2}))(\text{cont}_\lambda(i) - i)f_{tt}. \end{aligned}$$

By (7.16), we have

$$(a_{1,1} + 2a_{2,1} - a_{3,1} - 2a_{4,1}) - (a_{1,2} + 2a_{2,2} - a_{3,2} - 2a_{4,2}) = \begin{cases} 2, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 1, \\ 1, & \text{if } h_k(\mathbf{i}) = -1, \\ 0, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 0, \end{cases}$$

which implies

$$\left(\sum_{\ell \in A_{k,1}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,2}^i} y_\ell^\theta + \sum_{\ell \in A_{k,3}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,4}^i} y_\ell^\theta \right) f_{uu} = \begin{cases} 2(\text{cont}_\lambda(i) - i)f_{uu}, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 1, \\ (\text{cont}_\lambda(i) - i)f_{uu}, & \text{if } h_k(\mathbf{i}) = -1, \\ 0, & \text{if } h_k(\mathbf{i}) = 0 \text{ and } |\mathcal{A}\mathcal{R}_\lambda(i)| = 0. \end{cases} \quad (7.17)$$

By the construction, there exists $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$ if and only if $|\mathcal{A}\mathcal{R}_\lambda(i)| = 1$. Moreover, as $\mathbf{i} \in P_{k,+}^n$, we have $i_k = -\frac{1}{2}$ if $h_k(\mathbf{i}) = -1$ and $i_k \neq -\frac{1}{2}$ if $h_k(\mathbf{i}) = 0$. Therefore the Lemma follows by (7.17) and $c_t(k) = \text{cont}_\lambda(i_k)$. \square

Finally we prove the relation (3.20) when $\mathbf{i} \in P_{k,+}^n$.

7.33. Lemma. Suppose $1 \leq k \leq n - 1$ and $\mathbf{i} \in P_{k,+}^n$. Then for any $\mathbf{j}, \mathbf{k} \in I^n$, we have

$$e(\mathbf{j})\epsilon_k e(\mathbf{i})\epsilon_k e(\mathbf{k}) = (-1)^{a_k(\mathbf{i})}(1 + \delta_{i_k, -\frac{1}{2}})\left(\sum_{\ell \in A_{k,1}^i} y_\ell - 2 \sum_{\ell \in A_{k,2}^i} y_\ell + \sum_{\ell \in A_{k,3}^i} y_\ell - 2 \sum_{\ell \in A_{k,4}^i} y_\ell\right)e(\mathbf{j})\epsilon_k e(\mathbf{k}).$$

Proof. Choose arbitrary $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j})$. If $u(k) + u(k+1) \neq 0$, by Lemma 6.18 we have

$$f_{uu}\epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = 0 = \left(\sum_{\ell \in A_{k,1}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,2}^i} y_\ell^\theta + \sum_{\ell \in A_{k,3}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,4}^i} y_\ell^\theta\right)f_{uu}\epsilon_k^\theta e(\mathbf{k})^\theta. \quad (7.18)$$

If $u(k) + u(k+1) = 0$, let $\lambda = u_{k-1}$. If there is no $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$, we have $|\mathcal{A}\mathcal{R}_\lambda(i_k)| = 0$, which forces $h_k(\mathbf{i}) = 0$ by Lemma 3.4. Hence,

$$f_{uu}\epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = 0 = \left(\sum_{\ell \in A_{k,1}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,2}^i} y_\ell^\theta + \sum_{\ell \in A_{k,3}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,4}^i} y_\ell^\theta\right)f_{uu}\epsilon_k^\theta e(\mathbf{k})^\theta, \quad (7.19)$$

by Lemma 7.28.

Suppose there exist $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t} \stackrel{k}{\sim} \mathbf{u}$. Because $\mathbf{i} \in P_{k,+}^n$ and $\mathbf{i} \in I^n$, we have $h_k(\mathbf{i}) = 0$ or -1 . Hence, by (3.5) and (3.6), we always have $|\mathcal{A}\mathcal{R}_\lambda(i_k)| = 1$. Therefore, \mathbf{t} is unique, and we have

$$f_{uu}\epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = (-1)^{a_k(\mathbf{i})}2(c_t(k) - i_k)f_{uu}\epsilon_k^\theta e(\mathbf{k})^\theta,$$

by Lemma 7.4 and Lemma 6.12. So, we have

$$f_{uu}\epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = (-1)^{a_k(\mathbf{i})}(1 + \delta_{i_k, -\frac{1}{2}})\left(\sum_{\ell \in A_{k,1}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,2}^i} y_\ell^\theta + \sum_{\ell \in A_{k,3}^i} y_\ell^\theta - 2 \sum_{\ell \in A_{k,4}^i} y_\ell^\theta\right)f_{uu}\epsilon_k^\theta e(\mathbf{k})^\theta, \quad (7.20)$$

by Lemma 7.28.

Combining (7.18), (7.19) and (7.20), we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta = (-1)^{a_k(\mathbf{i})} (1 + \delta_{i_k, -\frac{1}{2}}) \left(\sum_{\ell \in A_{k,1}^1} y_\ell^\theta - 2 \sum_{\ell \in A_{k,2}^1} y_\ell^\theta + \sum_{\ell \in A_{k,3}^1} y_\ell^\theta - 2 \sum_{\ell \in A_{k,4}^1} y_\ell^\theta \right) e(\mathbf{j})^\theta \epsilon_k^\theta e(\mathbf{k})^\theta,$$

which completes the proof by lifting the elements into $\mathcal{B}_n(\delta)$. \square

We have proved that (3.15) - (3.19) hold by Lemma 7.12 and Lemma 7.18 - 7.23; and (3.20) holds by Lemma 7.27, Lemma 7.24 and Lemma 7.33. Therefore, the essential idempotent relations hold in $\mathcal{B}_n(\delta)$.

7.34. Proposition. *In $\mathcal{B}_n(\delta)$, the essential idempotent relations hold.*

We close this subsection by proving the untwist relations. We remind the readers that in the rest of this paper we assume $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I^n$.

7.35. Proposition. *In $\mathcal{B}_n(\delta)$, the untwist relation holds.*

Proof. We need to show that for any $1 \leq k \leq n-1$ and $\mathbf{i}, \mathbf{j} \in I^n$, we have

$$\begin{aligned} e(\mathbf{i}) \psi_k \epsilon_k e(\mathbf{j}) &= \begin{cases} (-1)^{a_k(\mathbf{i})} e(\mathbf{i}) \epsilon_k e(\mathbf{j}), & \text{if } \mathbf{i} \in I_{k,+}^n \text{ and } i_k \neq -\frac{1}{2}, \\ 0, & \text{otherwise;} \end{cases} \\ e(\mathbf{j}) \epsilon_k \psi_k e(\mathbf{i}) &= \begin{cases} (-1)^{a_k(\mathbf{i})} e(\mathbf{j}) \epsilon_k e(\mathbf{i}), & \text{if } \mathbf{i} \in I_{k,+}^n \text{ and } i_k \neq -\frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

By Proposition 7.6, we assume $i_k + i_{k+1} = 0$. Otherwise both sides of the equality will be 0. We will only prove the first equality, and the second equality follows by the similar argument.

Suppose $\mathbf{i} \in I_{k,+}^n$ and $i_k \neq 0, -\frac{1}{2}$. Choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. If $c_t(k) + c_t(k+1) \neq 0$, we have

$$f_{\mathbf{t}} \psi_k^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = 0 = (-1)^{a_k(\mathbf{i})} f_{\mathbf{t}} \epsilon_k^\theta e(\mathbf{j})^\theta, \quad (7.21)$$

by Lemma 7.2. If $c_t(k) + c_t(k+1) = 0$, because $i_k \neq 0$ and $\mathbf{i} \in I_{k,+}^n$, we have $\mathbf{i} \cdot s_k \neq \mathbf{i}$ and $h_k(\mathbf{i}) = 0$. Therefore, by Lemma 3.10, there exists a unique $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ such that $\mathbf{u} \stackrel{k}{\sim} \mathbf{t}$ and $\mathbf{u} \neq \mathbf{t}$, and $c_t(k) - i_k = c_u(k) - i_{k+1}$. We note that $c_t(k) - i_k = c_u(k) - i_{k+1}$ implies $c_t(k) + c_u(k) = 2(c_u(k) - i_{k+1})$ because $i_k + i_{k+1} = 0$. Hence, by Theorem 2.18, Lemma 7.4 and Lemma 6.12, we have

$$\begin{aligned} f_{\mathbf{t}} \psi_k^\theta \epsilon_k^\theta e(\mathbf{j})^\theta &= \sum_{\substack{\mathbf{v} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{v} \stackrel{k}{\sim} \mathbf{t}}} P_k(\mathbf{t})^{-1} s_k(\mathbf{t}, \mathbf{u}) Q_k(\mathbf{u})^{-1} P_k(\mathbf{u})^{-1} e_k(\mathbf{u}, \mathbf{v}) Q_k(\mathbf{v})^{-1} f_{\mathbf{v}} \\ &= \sum_{\substack{\mathbf{v} \in \mathcal{T}_n^{ud}(\mathbf{j}) \\ \mathbf{v} \stackrel{k}{\sim} \mathbf{t}}} \frac{P_k(\mathbf{u})^{-1} Q_k(\mathbf{u})^{-1} e_k(\mathbf{u}, \mathbf{u})}{c_t(k) + c_u(k)} P_k(\mathbf{t})^{-1} e_k(\mathbf{t}, \mathbf{v}) Q_k(\mathbf{v})^{-1} f_{\mathbf{v}} \\ &= \frac{P_k(\mathbf{u})^{-1} Q_k(\mathbf{u})^{-1} e_k(\mathbf{u}, \mathbf{u})}{2(c_u(k) - i_{k+1})} f_{\mathbf{t}} \epsilon_k^\theta e(\mathbf{j})^\theta \\ &= (-1)^{a_k(\mathbf{i})} f_{\mathbf{t}} \epsilon_k^\theta e(\mathbf{j})^\theta. \end{aligned} \quad (7.22)$$

Because \mathbf{t} is chosen arbitrary, by (7.21), (7.22) and Lemma 6.1, we have

$$e(\mathbf{i})^\theta \psi_k^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = (-1)^{a_k(\mathbf{i})} e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x),$$

and therefore $e(\mathbf{i}) \psi_k \epsilon_k e(\mathbf{j}) = e(\mathbf{i}) \epsilon_k e(\mathbf{j})$ by lifting the elements into $\mathcal{B}_n(\delta)$.

For the rest of the cases, if $h_k(\mathbf{i}) \neq 0$, we have $\mathbf{i} \cdot s_k \notin I^n$ by Lemma 3.20, which yields $e(\mathbf{i}) = 0$ by Proposition 2.19. Therefore, we have $e(\mathbf{i}) \psi_k \epsilon_k e(\mathbf{j}) = 0$ by Proposition 7.11. If $h_k(\mathbf{i}) = 0$, we have $i_k = i_{k+1} = 0$, which yields $e(\mathbf{i}) \psi_k = 0$ by Lemma 7.13. Therefore, we have $e(\mathbf{i}) \psi_k \epsilon_k e(\mathbf{j}) = 0$. \square

7.4. Tangle relations

In this subsection, we will prove the tangle relations hold in $\mathcal{B}_n^\theta(x)$ and extend to $\mathcal{B}_n(\delta)$ by lifting the elements into $\mathcal{B}_n(\delta)$. First we prove (3.24).

7.36. Lemma. *Suppose $1 \leq k \leq n-1$ and $\mathbf{i}, \mathbf{j} \in I^n$. Then we have $e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{j})^\theta (y_k^\theta + y_{k+1}^\theta) = 0$.*

Proof. By Lemma 6.18, we have $e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = 0$ if $i_k + i_{k+1} \neq 0$ or $j_k + j_{k+1} \neq 0$, where the Lemma holds. Suppose $i_k + i_{k+1} = j_k + j_{k+1} = 0$, as $\mathcal{L}_n(\mathcal{O})$ is a commutative subalgebra of $\mathcal{B}_n^\theta(x)$ and $Q_k^\theta(\mathbf{j})^{-1}e(\mathbf{j})^\theta \in \mathcal{L}_n(\mathcal{O})$, we have

$$e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{j})^\theta (y_k^\theta + y_{k+1}^\theta) = e(\mathbf{i})^\theta P_k^\theta(\mathbf{i})^{-1} e_k^\theta (L_k^\theta + L_{k+1}^\theta) Q_k^\theta(\mathbf{j})^{-1} e(\mathbf{j})^\theta = 0,$$

which completes the proof. \square

7.37. Lemma. Suppose $1 \leq k \leq n-1$ and $\mathbf{i}, \mathbf{j} \in I^n$. Then we have

$$e(\mathbf{i})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x), \quad \text{and} \quad e(\mathbf{i})^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta e(\mathbf{j})^\theta = e(\mathbf{i})^\theta \epsilon_{k-1}^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x).$$

Proof. We only prove the first equality. The second equality follows by the same argument. Choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. When $t(k)+t(k+1) \neq 0$, the equality holds because both sides of the equality are 0. When $t(k)+t(k+1) = 0$, we write $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$ and define

$$\mathbf{s} = (\alpha_1, \dots, \alpha_{k-2}, \alpha_{k-1}, -\alpha_{k-1}, \alpha_{k-1}, \alpha_{k+2}, \alpha_{k+3}, \dots, \alpha_n).$$

Because $\alpha_k + \alpha_{k+1} = 0$, \mathbf{s} is an up-down tableau. By Lemma 6.10 and Lemma 7.4, we have

$$f_{\mathbf{t}} \epsilon_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = f_{\mathbf{t}} \epsilon_k^\theta \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \epsilon_{k-1}^\theta \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \epsilon_k^\theta e(\mathbf{j})^\theta = f_{\mathbf{t}} P_k^\theta(\mathbf{i})^{-1} e_k^\theta \epsilon_{k-1}^\theta e_k^\theta Q_k^\theta(\mathbf{j})^{-1} e(\mathbf{j})^\theta = f_{\mathbf{t}} \epsilon_k^\theta e(\mathbf{j})^\theta.$$

As \mathbf{t} is chosen arbitrary, by Lemma 6.1, we have $e(\mathbf{i})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = e(\mathbf{i})^\theta \epsilon_k^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x)$. \square

Now we prove (3.23) hold in $\mathcal{B}_n^\theta(x)$. We separate the question into several cases based on the values of i_{k-1}, i_k and i_{k+1} . In more details, we consider the following three cases:

(6.4.1). When $i_{k-1} + i_{k+1} \neq 0$.

(6.4.2). When $i_{k-1} + i_{k+1} = 0$, $i_k + i_{k-1} \neq 0$ and $i_k + i_{k+1} \neq 0$.

(6.4.3). When $i_{k-1} + i_{k+1} = 0$, and $i_k + i_{k-1} = 0$ or $i_k + i_{k+1} = 0$.

First we consider the case (6.4.1).

7.38. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} + i_{k+1} \neq 0$. Then we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta = 0 = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta, \quad \text{and} \quad e(\mathbf{i})^\theta \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = 0 = e(\mathbf{i})^\theta \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta.$$

Proof. By Proposition 7.11, we have

$$\begin{aligned} e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta &= e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta e(\mathbf{i} \cdot s_k)^\theta \psi_k^\theta e(\mathbf{i})^\theta, \\ e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta &= e(\mathbf{j})^\theta \epsilon_k^\theta e(\mathbf{i} \cdot s_{k-1})^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta. \end{aligned}$$

By Lemma 6.18, both equalities equal to 0, which proves $e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta = 0 = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta$. Following the same argument, we can prove $e(\mathbf{i})^\theta \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = 0 = e(\mathbf{i})^\theta \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta$. \square

Then we consider the case (6.4.2). In this case, for any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $t(k-1)+t(k) \neq 0$ and $t(k)+t(k+1) \neq 0$. Hence, by Lemma 7.2, the actions of ψ_k^θ and ψ_{k-1}^θ on $f_{\mathbf{t}}$ are the same as in the KLR algebras. Therefore, for $\mathbf{i} \in I^n$ with $i_k + i_{k+1} \neq 0$, we have the following Lemma, which is analogue to [9, Definition 4.14, Lemma 4.18].

Recall that when $i_k \neq i_{k+1} + 1$, by Lemma 6.2, we have

$$\frac{1}{1 - L_k^\theta + L_{k+1}^\theta} e(\mathbf{i})^\theta = e(\mathbf{i})^\theta \frac{1}{1 - L_k^\theta + L_{k+1}^\theta} = \sum_{\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})} \frac{1}{1 - c_{\mathbf{t}}(k) + c_{\mathbf{t}}(k+1)} \frac{f_{\mathbf{t}}}{\gamma_{\mathbf{t}}} \in \mathcal{L}_n(\mathcal{O}).$$

7.39. Lemma. Suppose $\mathbf{i} \in I^n$ with $i_k + i_{k+1} \neq 0$. Then

$$\begin{aligned} e(\mathbf{i})^\theta \psi_k^\theta &= \begin{cases} e(\mathbf{i})^\theta \frac{1}{1 - L_{k+1}^\theta + L_k^\theta} (s_k^\theta - 1), & \text{if } i_k = i_{k+1}, \\ e(\mathbf{i})^\theta ((L_{k+1}^\theta - L_k^\theta) s_k^\theta - 1), & \text{if } i_k = i_{k+1} + 1, \\ e(\mathbf{i})^\theta \frac{1}{1 - L_{k+1}^\theta + L_k^\theta} ((L_{k+1}^\theta - L_k^\theta) s_k^\theta - 1), & \text{otherwise;} \end{cases} \\ \psi_k^\theta e(\mathbf{i})^\theta &= \begin{cases} (s_k^\theta + 1) \frac{1}{1 - L_k^\theta + L_{k+1}^\theta} e(\mathbf{i})^\theta, & \text{if } i_k = i_{k+1}, \\ (s_k^\theta (L_k^\theta - L_{k+1}^\theta) + 1) e(\mathbf{i})^\theta, & \text{if } i_k = i_{k+1} + 1, \\ (s_k^\theta (L_k^\theta - L_{k+1}^\theta) + 1) \frac{1}{1 - L_k^\theta + L_{k+1}^\theta} e(\mathbf{i})^\theta, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. It suffices to prove that for any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have

$$f_{\mathbf{tt}}\psi_k^\mathcal{O} = \begin{cases} f_{\mathbf{tt}}e(\mathbf{i})^\mathcal{O} \frac{1}{1-L_{k+1}^\mathcal{O}+L_k^\mathcal{O}}(s_k^\mathcal{O}-1), & \text{if } i_k = i_{k+1}, \\ f_{\mathbf{tt}}((L_{k+1}^\mathcal{O}-L_k^\mathcal{O})s_k^\mathcal{O}-1), & \text{if } i_k = i_{k+1}+1, \\ f_{\mathbf{tt}}e(\mathbf{i})^\mathcal{O} \frac{1}{1-L_{k+1}^\mathcal{O}+L_k^\mathcal{O}}((L_{k+1}^\mathcal{O}-L_k^\mathcal{O})s_k^\mathcal{O}-1), & \text{otherwise;} \end{cases} \quad (7.23)$$

$$\psi_k^\mathcal{O}f_{\mathbf{tt}} = \begin{cases} (s_k^\mathcal{O}+1)\frac{1}{1-L_k^\mathcal{O}+L_{k+1}^\mathcal{O}}e(\mathbf{i})^\mathcal{O}f_{\mathbf{tt}}, & \text{if } i_k = i_{k+1}, \\ (s_k^\mathcal{O}(L_k^\mathcal{O}-L_{k+1}^\mathcal{O})+1)f_{\mathbf{tt}}, & \text{if } i_k = i_{k+1}+1, \\ (s_k^\mathcal{O}(L_k^\mathcal{O}-L_{k+1}^\mathcal{O})+1)\frac{1}{1-L_k^\mathcal{O}+L_{k+1}^\mathcal{O}}e(\mathbf{i})^\mathcal{O}f_{\mathbf{tt}}, & \text{otherwise.} \end{cases} \quad (7.24)$$

Because $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and $i_k + i_{k+1} \neq 0$, we have $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$.

If $i_k = i_{k+1}$, because $i_k + i_{k+1} \neq 0$, we have $i_k = i_{k+1} \neq 0$. Hence, by the definition of h_k , we have $h_{k+1}(\mathbf{i}) = h_k(\mathbf{i}) + 2$ when $i_k \neq \pm \frac{1}{2}$, and $h_{k+1}(\mathbf{i}) = h_k(\mathbf{i}) + 3$ when $i_k = \pm \frac{1}{2}$. By Lemma 3.6, we have $-2 \leq h_k(\mathbf{i}), h_{k+1}(\mathbf{i}) \leq 0$, which forces $h_k(\mathbf{i}) = -2$ and $i_k \neq \pm \frac{1}{2}$. By Lemma 3.9, we have $\mathbf{t}(k) > 0$ and $\mathbf{t}(k+1) < 0$, or $\mathbf{t}(k) < 0$ and $\mathbf{t}(k+1) > 0$. Because $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$, by Lemma 2.6, we have $\mathbf{s} = \mathbf{t} \cdot s_k \in \mathcal{T}_n^{ud}(\mathbf{i})$. Then, by Theorem 2.18 and Lemma 7.2, we have

$$f_{\mathbf{tt}}\psi_k^\mathcal{O} = \frac{1}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}f_{\mathbf{tt}} + \frac{s_k(\mathbf{t}, \mathbf{s})}{1 - c_{\mathbf{s}}(k) + c_{\mathbf{s}}(k+1)}f_{\mathbf{st}},$$

and because $\mathbf{s} = \mathbf{t} \cdot s_k$, we have $c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k) = c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k+1)$. Hence,

$$\begin{aligned} f_{\mathbf{tt}}e(\mathbf{i})^\mathcal{O} \frac{1}{1-L_{k+1}^\mathcal{O}+L_k^\mathcal{O}}(s_k^\mathcal{O}-1) &= \frac{1}{1 - c_{\mathbf{t}}(k+1) + c_{\mathbf{t}}(k)} \left(\frac{1}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}f_{\mathbf{tt}} + s_k(\mathbf{t}, \mathbf{s})f_{\mathbf{st}} - 1 \right) \\ &= \frac{1}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}f_{\mathbf{tt}} + \frac{s_k(\mathbf{t}, \mathbf{s})}{1 - c_{\mathbf{s}}(k) + c_{\mathbf{s}}(k+1)}f_{\mathbf{st}}, \end{aligned}$$

which proves (7.23) when $i_k = i_{k+1}$.

If $i_k = i_{k+1} + 1$, when $\mathbf{t} \cdot s_k$ does not exist, by Theorem 2.18 and Lemma 7.2, we have

$$f_{\mathbf{tt}}((L_{k+1}^\mathcal{O}-L_k^\mathcal{O})s_k^\mathcal{O}-1) = \left(\frac{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)} - 1 \right)f_{\mathbf{tt}} = 0 = f_{\mathbf{tt}}\psi_k^\mathcal{O}.$$

When $\mathbf{s} = \mathbf{t} \cdot s_k$ is an up-down tableau, by Theorem 2.18 and Lemma 7.2, we have

$$f_{\mathbf{tt}}\psi_k^\mathcal{O} = s_k(\mathbf{t}, \mathbf{s})(c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k+1))f_{\mathbf{ts}},$$

and because $\mathbf{s} = \mathbf{t} \cdot s_k$, we have $c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k) = c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k+1)$. Hence,

$$\begin{aligned} f_{\mathbf{tt}}((L_{k+1}^\mathcal{O}-L_k^\mathcal{O})s_k^\mathcal{O}-1) &= \left(\frac{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)} - 1 \right)f_{\mathbf{tt}} + (c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k))s_k(\mathbf{t}, \mathbf{s})f_{\mathbf{ts}} \\ &= s_k(\mathbf{t}, \mathbf{s})(c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k+1))f_{\mathbf{ts}}, \end{aligned}$$

which proves (7.23) when $i_k = i_{k+1} + 1$.

For the other cases, when $\mathbf{t} \cdot s_k$ does not exist, by Theorem 2.18 and Lemma 7.2, we have

$$f_{\mathbf{tt}}e(\mathbf{i})^\mathcal{O} \frac{1}{1-L_{k+1}^\mathcal{O}+L_k^\mathcal{O}}((L_{k+1}^\mathcal{O}-L_k^\mathcal{O})s_k^\mathcal{O}-1) = \frac{1}{1 - c_{\mathbf{t}}(k+1) + c_{\mathbf{t}}(k)} \left(\frac{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)} - 1 \right)f_{\mathbf{tt}} = 0 = f_{\mathbf{tt}}\psi_k^\mathcal{O}.$$

When $\mathbf{s} = \mathbf{t} \cdot s_k$ is an up-down tableau, by Theorem 2.18 and Lemma 7.2, we have

$$f_{\mathbf{tt}}\psi_k^\mathcal{O} = s_k(\mathbf{t}, \mathbf{s})\frac{c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k+1)}{1 - c_{\mathbf{s}}(k) + c_{\mathbf{s}}(k+1)}f_{\mathbf{ts}},$$

and because $\mathbf{s} = \mathbf{t} \cdot s_k$, we have $c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k) = c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k+1)$. Hence,

$$\begin{aligned} &f_{\mathbf{tt}}e(\mathbf{i})^\mathcal{O} \frac{1}{1-L_{k+1}^\mathcal{O}+L_k^\mathcal{O}}((L_{k+1}^\mathcal{O}-L_k^\mathcal{O})s_k^\mathcal{O}-1) \\ &= \frac{1}{1 - c_{\mathbf{t}}(k+1) + c_{\mathbf{t}}(k)} \left(\frac{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)} - 1 \right)f_{\mathbf{tt}} + \frac{c_{\mathbf{t}}(k+1) - c_{\mathbf{t}}(k)}{1 - c_{\mathbf{t}}(k+1) + c_{\mathbf{t}}(k)}s_k(\mathbf{t}, \mathbf{s})f_{\mathbf{ts}} \\ &= s_k(\mathbf{t}, \mathbf{s})\frac{c_{\mathbf{s}}(k) - c_{\mathbf{s}}(k+1)}{1 - c_{\mathbf{s}}(k) + c_{\mathbf{s}}(k+1)}f_{\mathbf{ts}}, \end{aligned}$$

which proves (7.23) for the rest of the cases. Hence, (7.23) holds. Following the similar argument, (7.24) holds. \square

The symmetric group \mathfrak{S}_n acts from left on the rational functions $f \in R(L_1^\theta, \dots, L_n^\theta)$ by permuting variables. We denote $s_k \cdot f$ by $s_k^\theta f$.

By Lemma 7.39, for $\mathbf{i} \in I^n$ with $i_k + i_{k+1} \neq 0$, we define

$$\begin{aligned} M_k^\theta(\mathbf{i}) &= \begin{cases} \frac{1}{1-L_k^\theta+L_{k+1}^\theta} e(\mathbf{i})^\theta, & \text{if } i_k = i_{k+1}, \\ L_k^\theta - L_{k+1}^\theta, & \text{if } i_k = i_{k+1} + 1, \\ \frac{L_k^\theta - L_{k+1}^\theta}{1-L_k^\theta+L_{k+1}^\theta}, & \text{otherwise;} \end{cases} & N_k^\theta(\mathbf{i}) &= \begin{cases} \frac{1}{1-L_k^\theta+L_{k+1}^\theta} e(\mathbf{i})^\theta, & \text{if } i_k = i_{k+1}, \\ 1, & \text{if } i_k = i_{k+1} + 1, \\ \frac{1}{1-L_k^\theta+L_{k+1}^\theta}, & \text{otherwise;} \end{cases} \\ \tilde{M}_k^\theta(\mathbf{i}) &= \begin{cases} \frac{1}{1-L_{k+1}^\theta+L_k^\theta} e(\mathbf{i})^\theta, & \text{if } i_k = i_{k+1}, \\ L_{k+1}^\theta - L_k^\theta, & \text{if } i_k = i_{k+1} + 1, \\ \frac{L_{k+1}^\theta - L_k^\theta}{1-L_{k+1}^\theta+L_k^\theta}, & \text{otherwise;} \end{cases} & \tilde{N}_k^\theta(\mathbf{i}) &= \begin{cases} -\frac{1}{1-L_{k+1}^\theta+L_k^\theta} e(\mathbf{i})^\theta, & \text{if } i_k = i_{k+1}, \\ -1, & \text{if } i_k = i_{k+1} + 1, \\ -\frac{1}{1-L_{k+1}^\theta+L_k^\theta}, & \text{otherwise,} \end{cases} \end{aligned} \quad (7.25)$$

such that

$$\begin{aligned} \psi_k^\theta e(\mathbf{i})^\theta &= s_k^\theta M_k^\theta(\mathbf{i}) e(\mathbf{i})^\theta + N_k^\theta(\mathbf{i}) e(\mathbf{i})^\theta, \\ e(\mathbf{i})^\theta \psi_k^\theta &= e(\mathbf{i})^\theta \tilde{M}_k^\theta(\mathbf{i}) s_k^\theta + e(\mathbf{i})^\theta \tilde{N}_k^\theta(\mathbf{i}). \end{aligned} \quad (7.26)$$

The following is the technical result we need later.

7.40. Lemma. Suppose $\mathbf{i} \in I^n$ with $i_{k-1} + i_{k+1} = 0$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t}(k-1) + \mathbf{t}(k+1) = 0$ and $\mathbf{t}(k) \neq \mathbf{t}(k-1), \mathbf{t}(k+1)$, we have

$$\begin{aligned} M_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}} &= M_{k-1}^\theta(\mathbf{i})^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}}, \\ f_{\mathbf{t}} P_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} \tilde{M}_k^\theta(\mathbf{i}) &= f_{\mathbf{t}}^{s_{k-1}} P_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} \tilde{M}_{k-1}^\theta(\mathbf{i}). \end{aligned}$$

Proof. We only prove the first equality. The second equality follows by the similar method.

Because $\mathbf{t}(k-1) + \mathbf{t}(k+1) = 0$, we have $L_{k-1}^\theta f_{\mathbf{t}} = -L_{k+1}^\theta f_{\mathbf{t}}$. Moreover, if $i_{k-1} = i_k = i_{k+1}$, as $i_{k-1} + i_{k+1} = 0$, we have $i_{k-1} = i_k = i_{k+1} = 0$, which forces $\mathbf{t}(k-1) = -\mathbf{t}(k) = \mathbf{t}(k+1)$ by the construction of up-down tableaux. Hence, $i_{k-1} = i_k = i_{k+1}$ is excluded.

By the definition of $Q_k^\theta(\mathbf{i})$ and $i_{k-1} = -i_{k+1}$, we have

$$s_{k-1}^\theta Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}} = \begin{cases} \frac{1-L_{k-1}^\theta+L_k^\theta}{1-L_k^\theta+L_{k+1}^\theta} (L_k^\theta - L_{k+1}^\theta) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}, & \text{if } i_{k-1} = i_k, \\ \frac{1-L_{k-1}^\theta+L_k^\theta}{1-L_k^\theta+L_{k+1}^\theta} \frac{1}{L_{k-1}^\theta - L_k^\theta} Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}, & \text{if } i_k = i_{k+1}, \\ \frac{1}{1-L_k^\theta+L_{k+1}^\theta} \frac{L_k^\theta - L_{k+1}^\theta}{L_{k-1}^\theta - L_k^\theta} Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}, & \text{if } i_{k-1} = i_k + 1, \\ (1 - L_{k-1}^\theta + L_k^\theta) \frac{L_k^\theta - L_{k+1}^\theta}{L_{k-1}^\theta - L_k^\theta} Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}, & \text{if } i_k = i_{k+1} + 1, \\ \frac{L_k^\theta - L_{k+1}^\theta}{L_{k-1}^\theta - L_k^\theta} Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}, & \text{if } i_{k-1} = i_k + 1 \text{ and } i_k = i_{k+1} + 1, \\ \frac{1-L_{k-1}^\theta+L_k^\theta}{1-L_k^\theta+L_{k+1}^\theta} \frac{L_k^\theta - L_{k+1}^\theta}{L_{k-1}^\theta - L_k^\theta} Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}, & \text{otherwise.} \end{cases} \quad (7.27)$$

By (7.25) and (7.27), $M_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}} = M_{k-1}^\theta(\mathbf{i})^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}}$ can be verified by direct calculation. \square

Now we prove (3.23) when $i_{k-1} + i_{k+1} = 0$, $i_k + i_{k-1} \neq 0$ and $i_k + i_{k+1} \neq 0$.

7.41. Lemma. Suppose $1 < k < n$ and $\mathbf{i}, \mathbf{j} \in I^n$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t}(k-1) + \mathbf{t}(k+1) \neq 0$, $\mathbf{t}(k-1) + \mathbf{t}(k) \neq 0$ and $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$, we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{\mathbf{t}} = 0 = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{\mathbf{t}}, \quad (7.28)$$

$$f_{\mathbf{t}} \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = 0 = f_{\mathbf{t}} \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta. \quad (7.29)$$

Proof. We only prove (7.28), and (7.29) can be proved following the similar argument.

When $\mathbf{t} \cdot s_k$ does not exist, by Lemma 7.2, we have $\psi_k^\theta f_{\mathbf{t}} = 0$, which implies $e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{\mathbf{t}} = 0$; and when $\mathbf{s} = \mathbf{t} \cdot s_k$ is an up-down tableau, we have $\mathbf{s}(k-1) + \mathbf{s}(k) \neq 0$, which implies $e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{\mathbf{t}} = 0$ by Lemma 7.2.

When $\mathbf{t} \cdot s_{k-1}$ does not exist, by Lemma 7.2, we have $\psi_{k-1}^\theta f_{\mathbf{t}} = 0$, which implies $e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{\mathbf{t}} = 0$; and when $\mathbf{s} = \mathbf{t} \cdot s_{k-1}$ is an up-down tableau, we have $\mathbf{s}(k) + \mathbf{s}(k+1) \neq 0$, which implies $e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{\mathbf{t}} = 0$ by Lemma 7.2. Hence, (7.28) follows. \square

7.42. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} + i_{k+1} = 0$. If $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t}(k-1) + \mathbf{t}(k) \neq 0$ and $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$, for any $\mathbf{j} \in I^n$, we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{\mathbf{t}} = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{\mathbf{t}}, \quad (7.30)$$

$$f_{\mathbf{t}} \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = f_{\mathbf{t}} \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta. \quad (7.31)$$

Proof. We only prove the equality (7.30). The equality (7.31) can be proved following the similar argument.

When $t(k-1) + t(k+1) \neq 0$, because $t(k-1) + t(k) \neq 0$ and $t(k) + t(k+1) \neq 0$, (7.30) holds by Lemma 7.41.

When $t(k-1) + t(k+1) = 0$, because $t(k-1) + t(k) \neq 0$ and $t(k) + t(k+1) \neq 0$, by Proposition 7.11, Lemma 6.10 and (7.26), we have

$$\begin{aligned} e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{\mathbf{t}} &= e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} \psi_k^\theta f_{\mathbf{t}} = e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta \psi_k^\theta Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}} \\ &= e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta s_k^\theta M_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}} + e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta N_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}. \end{aligned}$$

Because $t(k-1) + t(k) \neq 0$, we have $\epsilon_{k-1}^\theta f_{\mathbf{t}} = 0$ by Lemma 7.2. Hence, as $N_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} \in \mathcal{L}_n(\theta)$, by Lemma 7.1, we have

$$e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta N_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}} = 0,$$

which yields

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{\mathbf{t}} = e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta s_k^\theta M_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}}. \quad (7.32)$$

Because $t(k-1) + t(k) \neq 0$ and $t(k) + t(k+1) \neq 0$, by Proposition 7.11 and (7.26), we have

$$\begin{aligned} e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{\mathbf{t}} &= e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta \psi_{k-1}^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}} \\ &= e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta s_{k-1}^\theta M_{k-1}^\theta(\mathbf{i})^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}} + e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta N_{k-1}^\theta(\mathbf{i})^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}}. \end{aligned}$$

Because $t(k) + t(k+1) \neq 0$, we have $e_k^\theta f_{\mathbf{t}} = 0$ by Lemma 7.2. Hence, by Proposition 7.14, we have

$$e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta N_{k-1}^\theta(\mathbf{i})^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}} = 0,$$

which yields

$$e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{\mathbf{t}} = e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta s_{k-1}^\theta M_{k-1}^\theta(\mathbf{i})^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}}. \quad (7.33)$$

By Lemma 7.40, we have $M_k^\theta(\mathbf{i}) Q_{k-1}^\theta(\mathbf{i} \cdot s_k)^{-1} f_{\mathbf{t}} = M_{k-1}^\theta(\mathbf{i})^{s_{k-1}} Q_k^\theta(\mathbf{i} \cdot s_{k-1})^{-1} f_{\mathbf{t}}$. Hence, because $e_k^\theta e_{k-1}^\theta s_k^\theta = e_k^\theta s_{k-1}^\theta$, the equality (7.30) holds by (7.32) and (7.33). \square

7.43. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} + i_{k+1} = 0$, $i_k + i_{k-1} \neq 0$ and $i_k + i_{k+1} \neq 0$. Then we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta \in \mathcal{B}_n^\theta(x), \quad \text{and} \quad e(\mathbf{i})^\theta \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = e(\mathbf{i})^\theta \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x).$$

Proof. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. As $i_k + i_{k-1} \neq 0$ and $i_k + i_{k+1} \neq 0$, we have $t(k) + t(k-1) \neq 0$ and $t(k) + t(k+1) \neq 0$. Hence, by Lemma 6.1 and Lemma 7.42, as \mathbf{t} is chosen arbitrary, we have

$$\begin{aligned} e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta &= e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta \in \mathcal{B}_n^\theta(x), \\ e(\mathbf{i})^\theta \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta &= e(\mathbf{i})^\theta \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x), \end{aligned}$$

which proves the Lemma. \square

It left us to consider the case (6.4.3).

7.44. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} + i_{k+1} = 0$, and $i_k + i_{k-1} = 0$ or $i_k + i_{k+1} = 0$. Then we have $i_k \neq \pm \frac{1}{2}$. Moreover, when $i_k = 0$, for any $\mathbf{j} \in I^n$, we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta = 0, \quad \text{and} \quad e(\mathbf{i})^\theta \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = e(\mathbf{i})^\theta \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = 0.$$

Proof. Suppose $i_k = \pm \frac{1}{2}$. By the assumption of the Lemma, we have $i_k = i_{k-1}$ or $i_k = i_{k+1}$. As $\mathbf{i} \in I^n$ and $i_k = \pm \frac{1}{2}$, by Lemma 3.7 we have $-2 \leq h_k(\mathbf{i}) \leq -1$. If $i_{k-1} = i_k$, then $h_{k-1}(\mathbf{i}) = h_k(\mathbf{i}) - 3 \leq -4$, which implies $\mathbf{i} \notin I^n$ by Lemma 3.6. If $i_{k+1} = i_k$, $h_{k+1}(\mathbf{i}) = h_k(\mathbf{i}) + 3 \geq 1$, which implies $\mathbf{i} \notin I^n$ by Lemma 3.6. Hence we have $\mathbf{i} \notin I^n$, which contradicts to the assumption of the Lemma. Hence we have $i_k \neq \pm \frac{1}{2}$.

If $i_k = 0$, then we have $i_{k-1} = i_k = i_{k+1} = 0$. Hence $e(\mathbf{i})^\theta \psi_k^\theta = e(\mathbf{i})^\theta \psi_{k-1}^\theta = 0$ by Lemma 7.13, which completes the proof. \square

7.45. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = i_k = -i_{k+1} \neq 0, \pm \frac{1}{2}$. Then for any $\mathbf{j} \in I^n$, we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta \in \mathcal{B}_n^\theta(x), \quad \text{and} \quad e(\mathbf{i})^\theta \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = e(\mathbf{i})^\theta \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x).$$

Proof. We only prove the first equality, and the second equality holds by using similar argument.

In order to prove $e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta$, it suffices to prove that for any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{\mathbf{t}} = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{\mathbf{t}}. \quad (7.34)$$

Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. As $i_{k-1} = i_k \neq 0, \pm \frac{1}{2}$, we have $i_{k-1} + i_k \neq 0$, which implies $t(k-1) + t(k) \neq 0$. By the construction of up-down tableaux, we have $t(k-1) \neq t(k)$, which yields that we will not have $t(k-1) + t(k+1) = t(k) + t(k+1) = 0$. Hence, we consider the following three cases.

Case 1: $t(k-1) + t(k+1) = 0$.

By the construction of up-down tableau, we have $t(k-1) \neq t(k)$, which implies $t(k) + t(k+1) \neq 0$. Moreover, as $i_{k-1} = i_k \neq 0$, we have $i_{k-1} + i_k \neq 0$, which implies $t(k-1) + t(k) \neq 0$. Hence, by Lemma 7.42, (7.34) holds when $t(k-1) + t(k+1) = 0$.

Case 2: $t(k) + t(k+1) = 0$.

In this case, we can write $t = (\alpha_1, \dots, \alpha_n)$ where $\alpha_k = -\alpha_{k+1}$. Define

$$s = (\alpha_1, \dots, \alpha_{k-1}, -\alpha_{k-1}, \alpha_{k-1}, \alpha_{k+2}, \dots, \alpha_n).$$

It is easy to see that s is an up-down tableau and we have $c_t(k-1) = -c_s(k)$. As $s \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$, by Lemma 3.20 we have $h_k(\mathbf{i}) = 0$, which implies that s is the unique up-down tableau in $\mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ such that $s \stackrel{k}{\sim} t$ by Lemma 3.10. By Theorem 2.18, Lemma 7.2, Lemma 7.4 and Lemma 6.10, we have

$$\begin{aligned} e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta f_{tt} &= e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \frac{f_{ss}}{\gamma_s} \psi_k^\theta f_{tt} = e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta \frac{f_{ss}}{\gamma_s} e_k^\theta \frac{Q_k^\theta(\mathbf{i})^{-1}}{c_t(k) + c_s(k)} f_{tt} \\ &= e(\mathbf{j})^\theta P_k^\theta(\mathbf{j})^{-1} e_k^\theta e_{k-1}^\theta e_k^\theta \frac{Q_k^\theta(\mathbf{i})^{-1}}{c_t(k) - c_t(k-1)} f_{tt} = \frac{1}{c_t(k) - c_t(k-1)} e(\mathbf{j})^\theta \epsilon_k^\theta f_{tt}, \end{aligned} \quad (7.35)$$

and by Lemma 7.2 and Lemma 7.4, we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta f_{tt} = e(\mathbf{j})^\theta \epsilon_k^\theta \frac{f_{tt}}{\gamma_t} \psi_{k-1}^\theta f_{tt} = \frac{1}{c_t(k) - c_t(k-1)} e(\mathbf{j})^\theta \epsilon_k^\theta f_{tt}. \quad (7.36)$$

Hence by (7.35) and (7.36), (7.34) holds when $t(k) + t(k+1) = 0$.

Case 3: $t(k-1) + t(k+1) \neq 0$ and $t(k) + t(k+1) \neq 0$.

By Lemma 7.41, (7.34) holds when $t(k-1) + t(k+1) \neq 0$ and $t(k) + t(k+1) \neq 0$. \square

The next Lemma can be proved using the same argument as Lemma 7.45.

7.46. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = -i_{k+1} \neq 0, \pm \frac{1}{2}$. Then for any $\mathbf{j} \in I^n$, we have

$$e(\mathbf{j})^\theta \epsilon_k^\theta \epsilon_{k-1}^\theta \psi_k^\theta e(\mathbf{i})^\theta = e(\mathbf{j})^\theta \epsilon_k^\theta \psi_{k-1}^\theta e(\mathbf{i})^\theta \in \mathcal{B}_n^\theta(x), \quad \text{and} \quad e(\mathbf{i})^\theta \psi_k^\theta \epsilon_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta = e(\mathbf{i})^\theta \psi_{k-1}^\theta \epsilon_k^\theta e(\mathbf{j})^\theta \in \mathcal{B}_n^\theta(x).$$

7.47. Lemma. In $\mathcal{B}_n^\theta(x)$, the tangle relations hold.

Proof. The relation (3.24) holds by Lemma 7.36 and Lemma 7.37; and the relation (3.23) holds by Lemma 7.38, Lemma 7.43, Lemma 7.44, Lemma 7.45 and Lemma 7.46. \square

7.48. Corollary. Suppose $1 \leq k < n$, we have $\epsilon_k^\theta \epsilon_{k+1}^\theta \psi_k^\theta = \epsilon_k^\theta \psi_{k+1}^\theta$, and $\psi_k^\theta \epsilon_{k+1}^\theta \epsilon_k^\theta = \psi_{k+1}^\theta \epsilon_k^\theta$.

Proof. By Lemma 7.47, we have $\epsilon_k^\theta \epsilon_{k+1}^\theta \psi_k^\theta = \epsilon_k^\theta \epsilon_{k+1}^\theta \epsilon_k^\theta \psi_{k+1}^\theta = \epsilon_k^\theta \psi_{k+1}^\theta$, and $\psi_k^\theta \epsilon_{k+1}^\theta \epsilon_k^\theta = \psi_{k+1}^\theta \epsilon_k^\theta \epsilon_{k+1}^\theta \epsilon_k^\theta = \psi_{k+1}^\theta \epsilon_k^\theta$, which completes the proof. \square

The next Proposition follows by Lemma 7.47.

7.49. Proposition. In $\mathcal{B}_n(\delta)$, the tangle relations hold.

7.5. Braid relations

In this subsection, we prove the braid relations hold in $\mathcal{B}_n(\delta)$. The braid relations are determined by the values of i_{k-1} , i_k and i_{k+1} . Hence, we separate the question into several cases. In more details, we consider the following cases:

(6.5.1). When $i_{k-1} + i_k \neq 0$, $i_{k-1} + i_{k+1} \neq 0$ and $i_k + i_{k+1} \neq 0$.

(6.5.2). When $i_{k-1} + i_k = 0$ and $i_{k+1} \neq \pm i_{k-1}$, or $i_{k-1} + i_{k+1} = 0$ and $i_k \neq \pm i_{k-1}$, or $i_k + i_{k+1} = 0$ and $i_{k-1} \neq \pm i_k$.

(6.5.3). When $i_{k-1} = i_k = -i_{k+1}$, or $i_{k-1} = -i_k = i_{k+1}$, or $-i_{k-1} = i_k = i_{k+1}$.

First we consider the case (6.5.1).

7.50. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfies (6.5.1). Then we have

$$e(\mathbf{i}) \psi_k \psi_{k-1} \psi_k = \begin{cases} e(\mathbf{i}) \psi_{k-1} \psi_k \psi_{k-1} - e(\mathbf{i}), & \text{if } i_{k-1} = i_{k+1} = i_k - 1, \\ e(\mathbf{i}) \psi_{k-1} \psi_k \psi_{k-1} + e(\mathbf{i}), & \text{if } i_{k-1} = i_{k+1} = i_k + 1, \\ e(\mathbf{i}) \psi_{k-1} \psi_k \psi_{k-1}, & \text{otherwise.} \end{cases}$$

Proof. Suppose $t \in \mathcal{T}_n^{ud}(\mathbf{i})$. Because $i_{k-1} + i_k \neq 0$, $i_{k-1} + i_{k+1} \neq 0$ and $i_k + i_{k+1} \neq 0$, we have $c_t(k-1) + c_t(k) \neq 0$, $c_t(k-1) + c_t(k+1) \neq 0$ and $c_t(k) + c_t(k+1) \neq 0$. By Corollary 7.3 the Lemma holds. \square

Next we consider the case (6.5.2). Note that when $i_k + i_{k+1} = 0$, $i_{k-1} \neq \pm i_k$ is equivalent to $i_{k-1} \neq i_k$ and $i_{k-1} \neq i_{k+1}$. We separate this case further. In more details, we consider the following cases:

(6.5.2.1). When $i_k + i_{k+1} = 0$, and $|i_{k-1} - i_k| > 1$ and $|i_{k-1} - i_{k+1}| > 1$.

(6.5.2.2). When $i_{k-1} + i_k = 0$, and $|i_{k+1} - i_{k-1}| > 1$ and $|i_{k+1} - i_k| > 1$.

(6.5.2.3). When $i_{k-1} + i_{k+1} = 0$, and $|i_k - i_{k-1}| > 1$ and $|i_k - i_{k+1}| > 1$.

(6.5.2.4). When $i_k + i_{k+1} = 0$, and $|i_{k-1} - i_k| = 1$, or $|i_{k-1} - i_{k+1}| = 1$, or $|i_{k-1} - i_k| = |i_{k-1} - i_{k+1}| = 1$.

(6.5.2.5). When $i_{k-1} + i_k = 0$, and $|i_{k+1} - i_{k-1}| = 1$, or $|i_{k+1} - i_k| = 1$, or $|i_{k+1} - i_{k-1}| = |i_{k+1} - i_k| = 1$.

(6.5.2.6). When $i_{k-1} + i_{k+1} = 0$, and $|i_k - i_{k-1}| = 1$, or $|i_k - i_{k+1}| = 1$, or $|i_k - i_{k-1}| = |i_k - i_{k+1}| = 1$.

It is easy to see that if $\mathbf{i} \in I^n$ satisfies (6.5.2), then \mathbf{i} satisfies one of (6.5.2.1) - (6.5.2.6). In more details, if \mathbf{i} satisfy (6.5.2), then we have $r, l \in \{k-1, k, k+1\}$ such that $i_r + i_l = 0$. Let $m \in \{k-1, k, k+1\}$ and $m \neq r, l$. If $|i_m - i_r| = 0$, then $i_m = i_r$, which contradicts to the fact \mathbf{i} satisfies (6.5.2). Similarly, if $|i_m - i_l| = 0$, it leads to contradiction. Hence, we have $|i_m - i_r| \geq 1$ and $|i_m - i_l| \geq 1$. This shows that \mathbf{i} satisfies one of (6.5.2.1) - (6.5.2.6).

First we prove (6.5.2.1) - (6.5.2.3). The following Lemmas prove (6.5.2.1).

7.51. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.1). Then we have $e(\mathbf{i})^\circ \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ = e(\mathbf{i})^\circ \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ$ if $i_k = i_{k+1} = 0$.

Proof. Let $\mathbf{j} = (j_1, \dots, j_n) = \mathbf{i} \cdot s_k s_{k-1} s_k$. By Proposition 7.11, we have

$$e(\mathbf{i})^\circ \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ = e(\mathbf{i})^\circ \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ e(\mathbf{j})^\circ, \quad \text{and} \quad e(\mathbf{i})^\circ \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ = e(\mathbf{i})^\circ \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ e(\mathbf{j})^\circ. \quad (7.37)$$

When $i_k = i_{k+1} = 0$, we have $j_{k-1} = j_k = 0$. Then by Lemma 7.13, we have $e(\mathbf{i})^\circ \psi_k^\circ = 0 = \psi_{k-1}^\circ e(\mathbf{j})^\circ$. Therefore, by (7.37), we have $e(\mathbf{i})^\circ \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ = 0 = e(\mathbf{i})^\circ \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ$, which proves the Lemma. \square

7.52. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.1). For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have

$$f_{\mathbf{t}} \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ = f_{\mathbf{t}} \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ.$$

Proof. Because $|i_{k-1} - i_k| > 1$, $|i_{k-1} - i_{k+1}| > 1$ and $i_k + i_{k+1} = 0$, we have $i_{k-1} + i_k \neq 0$ and $i_{k-1} + i_{k+1} \neq 0$. Therefore, we have $\mathbf{t}(k-1) + \mathbf{t}(k) \neq 0$ and $\mathbf{t}(k-1) + \mathbf{t}(k+1) \neq 0$. It is easy to see that when $\mathbf{t}(k) + \mathbf{t}(k+1) \neq 0$, the Lemma holds by Corollary 7.3. In the rest of the proof, we consider $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$.

Set $\mathbf{j} = \mathbf{i} \cdot s_k s_{k-1} s_k = \mathbf{i} \cdot s_{k-1} s_k s_{k-1}$. By Proposition 7.11, we have

$$f_{\mathbf{t}} \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ = e(\mathbf{i})^\circ \psi_k^\circ e(\mathbf{i} \cdot s_k)^\circ \psi_{k-1}^\circ e(\mathbf{i} \cdot s_k s_{k-1})^\circ \psi_k^\circ e(\mathbf{i} \cdot s_k s_{k-1} s_k)^\circ, \quad (7.38)$$

$$\text{and} \quad f_{\mathbf{t}} \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ = e(\mathbf{i})^\circ \psi_{k-1}^\circ e(\mathbf{i} \cdot s_{k-1})^\circ \psi_k^\circ e(\mathbf{i} \cdot s_{k-1} s_k)^\circ \psi_{k-1}^\circ e(\mathbf{i} \cdot s_{k-1} s_k s_{k-1})^\circ. \quad (7.39)$$

If $\mathbf{j} \notin I^n$, (7.38) and (7.39) both equal to 0 by Lemma 6.1. Hence we have

$$f_{\mathbf{t}} \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ = 0 = f_{\mathbf{t}} \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ,$$

where the Lemma holds. Hence, in the rest of the proof, we assume that $\mathbf{j} \in I^n$.

When $\mathbf{j} \in I^n$, as $j_{k-1} = i_{k+1} = -i_k$ and $|i_k - i_{k-1}| > 1$, by the definition of h_k , we have $h_{k-1}(\mathbf{j}) = -h_k(\mathbf{i})$. Because $\mathbf{i}, \mathbf{j} \in I^n$, by Lemma 3.6, it forces $h_{k-1}(\mathbf{j}) = h_k(\mathbf{i}) = 0$. Then by Lemma 3.20 and Lemma 3.21, we have $\mathbf{i} \cdot s_k, \mathbf{i} \cdot s_k s_{k-1}, \mathbf{i} \cdot s_{k-1}, \mathbf{i} \cdot s_{k-1} s_k \in I^n$.

Recall we have $\mathbf{t}(k) + \mathbf{t}(k+1) = 0$. Because we have $h_k(\mathbf{i}) = 0$, by Lemma 3.10, there exists a unique $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ such that $\mathbf{t} \stackrel{k}{\sim} \mathbf{s}$. As $i_k = -i_{k+1} \neq 0$, we have $\mathbf{s} \neq \mathbf{t}$.

Because $|i_{k-1} - i_k| > 1$, $|i_{k-1} - i_{k+1}| > 1$ and $i_k + i_{k+1} = 0$, we have $i_{k-1} + i_k \neq 0$ and $i_{k-1} + i_{k+1} \neq 0$. Hence, by Lemma 3.21, there exist $\mathbf{v} = \mathbf{s} \cdot s_{k-1} s_k \in \mathcal{T}_n^{ud}(\mathbf{j})$. As $h_{k-1}(\mathbf{j}) = 0$, by Lemma 3.10, there exists a unique $\mathbf{u} \in \mathcal{T}_n^{ud}(\mathbf{j} \cdot s_{k-1})$ such that $\mathbf{u} \stackrel{k-1}{\sim} \mathbf{v}$. Because of the uniqueness of \mathbf{u} , we have $\mathbf{u} = \mathbf{t} \cdot s_{k-1} s_k \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_{k-1} s_k)$.

Hence, by Lemma 7.2 and Lemma 7.4, we have

$$\begin{aligned} f_{\mathbf{t}} \psi_k^\circ \psi_{k-1}^\circ \psi_k^\circ &= f_{\mathbf{t}} \psi_k^\circ \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \psi_{k-1}^\circ \psi_k^\circ \frac{f_{\mathbf{vv}}}{\gamma_{\mathbf{v}}} = \frac{1}{c_{\mathbf{t}}(k) + c_{\mathbf{s}}(k)} f_{\mathbf{t}} \epsilon_k^\circ \frac{f_{\mathbf{ss}}}{\gamma_{\mathbf{s}}} \psi_{k-1}^\circ \psi_k^\circ \frac{f_{\mathbf{vv}}}{\gamma_{\mathbf{v}}} \\ &= \frac{1}{c_{\mathbf{t}}(k) + c_{\mathbf{s}}(k)} f_{\mathbf{t}} \epsilon_k^\circ \psi_{k-1}^\circ \psi_k^\circ \frac{f_{\mathbf{vv}}}{\gamma_{\mathbf{v}}}; \end{aligned} \quad (7.40)$$

$$\begin{aligned} f_{\mathbf{t}} \psi_{k-1}^\circ \psi_k^\circ \psi_{k-1}^\circ &= f_{\mathbf{t}} \psi_{k-1}^\circ \psi_k^\circ \frac{f_{\mathbf{uu}}}{\gamma_{\mathbf{u}}} \psi_{k-1}^\circ \frac{f_{\mathbf{vv}}}{\gamma_{\mathbf{v}}} = \frac{1}{c_{\mathbf{u}}(k-1) + c_{\mathbf{v}}(k-1)} f_{\mathbf{t}} \psi_{k-1}^\circ \psi_k^\circ \frac{f_{\mathbf{uu}}}{\gamma_{\mathbf{u}}} \epsilon_{k-1}^\circ \frac{f_{\mathbf{vv}}}{\gamma_{\mathbf{v}}} \\ &= \frac{1}{c_{\mathbf{u}}(k-1) + c_{\mathbf{v}}(k-1)} f_{\mathbf{t}} \psi_{k-1}^\circ \psi_k^\circ \epsilon_{k-1}^\circ \frac{f_{\mathbf{vv}}}{\gamma_{\mathbf{v}}}. \end{aligned} \quad (7.41)$$

By Lemma 7.47, Corollary 7.48 and Corollary 7.3, we have

$$\begin{aligned} f_{\mathbf{t}} \epsilon_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \frac{f_{\mathbf{v}}}{\gamma_{\mathbf{v}}} &= f_{\mathbf{t}} \epsilon_k^{\mathcal{O}} \epsilon_{k-1}^{\mathcal{O}} (\psi_k^{\mathcal{O}})^2 \frac{f_{\mathbf{v}}}{\gamma_{\mathbf{v}}} = f_{\mathbf{t}} \epsilon_k^{\mathcal{O}} \epsilon_{k-1}^{\mathcal{O}} \frac{f_{\mathbf{v}}}{\gamma_{\mathbf{v}}}, \\ f_{\mathbf{t}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \epsilon_{k-1}^{\mathcal{O}} \frac{f_{\mathbf{v}}}{\gamma_{\mathbf{v}}} &= f_{\mathbf{t}} (\psi_{k-1}^{\mathcal{O}})^2 \epsilon_k^{\mathcal{O}} \epsilon_{k-1}^{\mathcal{O}} \frac{f_{\mathbf{v}}}{\gamma_{\mathbf{v}}} = f_{\mathbf{t}} \epsilon_k^{\mathcal{O}} \epsilon_{k-1}^{\mathcal{O}} \frac{f_{\mathbf{v}}}{\gamma_{\mathbf{v}}}, \end{aligned}$$

which yields

$$f_{\mathbf{t}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} = \frac{c_u(k-1) + c_v(k-1)}{c_t(k) + c_s(k)} f_{\mathbf{t}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}}, \quad (7.42)$$

by (7.40) and (7.41). Because $\mathbf{u} = \mathbf{t} \cdot s_{k-1} s_k$, we have $c_t(k) = c_u(k-1)$; and because $\mathbf{v} = \mathbf{s} \cdot s_{k-1} s_k$, we have $c_s(k) = c_v(k-1)$. Therefore, by (7.42), we complete the proof. \square

7.53. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.1). Then we have $e(\mathbf{i})^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} = e(\mathbf{i})^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}}$.

Proof. Because $i_k + i_{k+1} = 0$, we have $i_k = -i_{k+1}$. By Lemma 7.51, the Lemma holds if $i_k = -i_{k+1} = 0$. When $i_k = -i_{k+1} \neq 0$, choose arbitrary $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, we have $f_{\mathbf{t}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} = f_{\mathbf{t}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}}$ by Lemma 7.52. As \mathbf{t} is chosen arbitrary, by Lemma 6.1, the Lemma follows. \square

The following Lemmas prove (6.5.2.2) and (6.5.2.3).

7.54. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.3). Then we have $e(\mathbf{i})^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} = e(\mathbf{i})^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}}$.

Proof. Set $\mathbf{j} = \mathbf{i} \cdot s_k s_{k-1} s_k$, $\mathbf{l} = \mathbf{i} \cdot s_{k-1}$ and $\mathbf{m} = \mathbf{j} \cdot s_k$. Then by Proposition 7.11 and Lemma 7.53, we have

$$\begin{aligned} e(\mathbf{l})^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} e(\mathbf{m})^{\mathcal{O}} &= e(\mathbf{l})^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} e(\mathbf{m})^{\mathcal{O}} \\ &= e(\mathbf{l})^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} e(\mathbf{m})^{\mathcal{O}} = e(\mathbf{l})^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} e(\mathbf{j})^{\mathcal{O}} \psi_k^{\mathcal{O}} e(\mathbf{m})^{\mathcal{O}}. \end{aligned} \quad (7.43)$$

Because $|i_k - i_{k-1}| > 1$, $|i_k - i_{k+1}| > 1$ and $i_{k-1} + i_{k+1} = 0$, we have $i_{k-1} + i_k \neq 0$ and $i_k + i_{k+1} \neq 0$. By Corollary 7.3, we have $(\psi_{k-1}^{\mathcal{O}})^2 e(\mathbf{i})^{\mathcal{O}} = e(\mathbf{i})^{\mathcal{O}}$ and $e(\mathbf{j})^{\mathcal{O}} (\psi_k^{\mathcal{O}})^2 = e(\mathbf{j})^{\mathcal{O}}$. Hence multiplying $\psi_{k-1}^{\mathcal{O}}$ from left and $\psi_k^{\mathcal{O}}$ from right on both sides of (7.43), the Lemma follows. \square

The next Lemma can be proved using the same method as Lemma 7.54.

7.55. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.2). Then we have $e(\mathbf{i})^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} = e(\mathbf{i})^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}}$.

Next we prove (6.5.2.4) - (6.5.2.6). We start by considering some special cases.

7.56. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.4). If $i_{k-1} = 0$, or $i_k = i_{k+1} = 0$, or $i_k = -i_{k+1} = \pm \frac{1}{2}$, we have

$$\begin{aligned} e(\mathbf{i}) \psi_k \psi_{k-1} \psi_k &= e(\mathbf{i}) \psi_{k-1} \psi_k \psi_{k-1} = e(\mathbf{i}) \epsilon_{k-1} e(\mathbf{i} \cdot s_k s_{k-1} s_k) = 0, \\ \psi_k \psi_{k-1} \psi_k e(\mathbf{i}) &= \psi_{k-1} \psi_k \psi_{k-1} e(\mathbf{i}) = e(\mathbf{i} \cdot s_k s_{k-1} s_k) \epsilon_{k-1} e(\mathbf{i}) = 0. \end{aligned}$$

Proof. Set $\mathbf{j} = \mathbf{i} \cdot s_k s_{k-1} s_k$. First we show that under the assumption of the Lemma, for any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$, we have $\mathbf{t}|_{k-2} \neq \mathbf{s}|_{k-2}$.

Suppose $i_{k-1} = 0$. By the assumption, we have $i_k = -i_{k+1} = \pm 1$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, as $i_{k-1} = 0$ and $i_k = -i_{k+1} = \pm 1$, by the construction of up-down tableaux, we have $\mathbf{t}(k-1), \mathbf{t}(k) > 0$ or $\mathbf{t}(k-1), \mathbf{t}(k) < 0$. If $\mathbf{t}(k-1), \mathbf{t}(k) > 0$ and $i_k = 1$, let $\lambda = \mathbf{t}_{k-2}$ and $(i, j) = \mathbf{t}(k-1)$. Then we have $\mathbf{t}(k) = (i, j+1)$, which implies $(i-1, j) \in [\lambda]$ and $(i-1, j) \notin \mathcal{R}(\lambda)$, and $(i+1, j) \notin [\lambda]$ and $(i+1, j) \notin \mathcal{A}(\lambda)$. Therefore, we have $\mathcal{A}_{\mathcal{R}(\lambda)}(-1) = \emptyset$. Hence, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$, we have $\mathbf{t}|_{k-2} \neq \mathbf{s}|_{k-2}$. Following the similar argument, if $\mathbf{t}(k-1), \mathbf{t}(k) < 0$, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$, we have $\mathbf{t}|_{k-2} \neq \mathbf{s}|_{k-2}$.

Suppose $i_k = i_{k+1} = 0$. By the assumption, we have $i_{k-1} = \pm 1$. For any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$, as $i_{k-1} = \pm 1$ and $i_k = 0$, by the construction of up-down tableaux, we have $\mathbf{t}(k-1), \mathbf{t}(k) > 0$ or $\mathbf{t}(k-1), \mathbf{t}(k) < 0$. If $\mathbf{t}(k-1), \mathbf{t}(k) > 0$ and $i_{k-1} = 1$, let $\lambda = \mathbf{t}_{k-2}$ and $(i, j) = \mathbf{t}(k-1)$. Then we have $\mathbf{t}(k) = (i+1, j)$, which implies $(i, j-1) \in [\lambda]$ and $(i, j-1) \notin \mathcal{R}(\lambda)$, and $(i+1, j) \notin \mathcal{A}(\lambda)$. Therefore, we have $\mathcal{A}_{\mathcal{R}(\lambda)}(0) = \emptyset$. Hence, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$, we have $\mathbf{t}|_{k-2} \neq \mathbf{s}|_{k-2}$. Following the similar argument, if $\mathbf{t}(k-1), \mathbf{t}(k) < 0$, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$, we have $\mathbf{t}|_{k-2} \neq \mathbf{s}|_{k-2}$.

Suppose $i_k = -i_{k+1} = \pm \frac{1}{2}$. Because $i_{k-1} \neq \pm i_k$ in (6.5.2), we have $i_{k-1} = \pm \frac{3}{2}$. Suppose $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$. Following the similar argument as above, by the construction of up-down tableaux, for any $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$, we have $\mathbf{t}|_{k-2} \neq \mathbf{s}|_{k-2}$.

Therefore, if $\mathbf{i} \in I^n$ satisfying the assumptions of the Lemma, for any $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ and $\mathbf{s} \in \mathcal{T}_n^{ud}(\mathbf{j})$, we have $\mathbf{t}|_{n-2} \neq \mathbf{s}|_{n-2}$. Hence, we have

$$\begin{aligned} e(\mathbf{i})^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} &= e(\mathbf{i})^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} = e(\mathbf{i})^{\mathcal{O}} \epsilon_k^{\mathcal{O}} \epsilon_{k-1}^{\mathcal{O}} e(\mathbf{i} \cdot s_k s_{k-1} s_k)^{\mathcal{O}} = 0 \in \mathcal{B}_n^{\mathcal{O}}(x), \\ \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} &= \psi_{k-1}^{\mathcal{O}} \psi_k^{\mathcal{O}} \psi_{k-1}^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} = e(\mathbf{i} \cdot s_k s_{k-1} s_k)^{\mathcal{O}} \epsilon_{k-1}^{\mathcal{O}} \epsilon_k^{\mathcal{O}} e(\mathbf{i})^{\mathcal{O}} = 0 \in \mathcal{B}_n^{\mathcal{O}}(x). \end{aligned}$$

The Lemma follows by lifting the elements into $\mathcal{B}_n(\delta)$. \square

The next Lemma can be proved by the same argument as Lemma 7.56.

7.57. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.6). If $i_k = 0$, or $i_{k-1} = i_{k+1} = 0$, or $i_{k-1} = -i_{k+1} = \pm \frac{1}{2}$, we have $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$.

7.58. Remark. Suppose $\mathbf{i} \in I^n$ satisfying (6.5.2.4) and $\mathbf{j} = \mathbf{i} \cdot s_k s_{k-1} s_k$. By the definition of h_k , as $i_k = -i_{k+1} = -j_{k-1}$, we have

$$h_k(\mathbf{i}) = -h_{k-1}(\mathbf{j}) + \delta_{i_k, -i_{k-1}+1} + \delta_{i_k, -i_{k-1}-1} + 2\delta_{i_k, i_{k-1}} - \delta_{i_k, i_{k-1}-1} - \delta_{i_k, i_{k-1}+1} - 2\delta_{i_k, -i_{k-1}}.$$

Because \mathbf{i} satisfies (6.5.2.4), we have $i_k \neq \pm i_{k-1}$, which implies $\delta_{i_k, i_{k-1}} = \delta_{i_k, -i_{k-1}} = 0$; and we have $|i_k - i_{k-1}| = 1$, or $|i_{k+1} - i_{k-1}| = |i_k + i_{k-1}| = 1$, or $|i_k - i_{k-1}| = |i_k + i_{k-1}| = 1$.

Assume \mathbf{i} is a residue sequence satisfies (6.5.2.4) and does not satisfy the assumption of Lemma 7.56. Because $i_k \neq 0$ and $i_{k-1} \neq 0$, we have $|i_k - i_{k-1}| \neq |i_k + i_{k-1}|$.

If $|i_k - i_{k-1}| = 1$, we have $i_k = i_{k-1} \pm 1$, which implies $\delta_{i_k, i_{k-1}-1} + \delta_{i_k, i_{k-1}+1} = 1$. It is easy to verify that $\delta_{i_k, -i_{k-1}+1} + \delta_{i_k, -i_{k-1}-1} \neq 0$ only if $i_{k-1} = 0$ or $i_k = 0$. Therefore, by excluding the assumptions of Lemma 7.56, we have

$$h_k(\mathbf{i}) = -h_{k-1}(\mathbf{j}) + \delta_{i_k, -i_{k-1}+1} + \delta_{i_k, -i_{k-1}-1} + 2\delta_{i_k, i_{k-1}} - \delta_{i_k, i_{k-1}-1} - \delta_{i_k, i_{k-1}+1} - 2\delta_{i_k, -i_{k-1}} = -h_{k-1}(\mathbf{j}) - 1,$$

when $|i_k - i_{k-1}| = 1$.

If $|i_k + i_{k-1}| = 1$, we have $i_k = -i_{k-1} \pm 1$, which implies $\delta_{i_k, -i_{k-1}+1} + \delta_{i_k, -i_{k-1}-1} = 1$. It is easy to verify that $\delta_{i_k, i_{k-1}-1} + \delta_{i_k, i_{k-1}+1} \neq 0$ only if $i_{k-1} = 0$ or $i_k = 0$. Therefore, by excluding the assumptions of Lemma 7.56, we have

$$h_k(\mathbf{i}) = -h_{k-1}(\mathbf{j}) + \delta_{i_k, -i_{k-1}+1} + \delta_{i_k, -i_{k-1}-1} + 2\delta_{i_k, i_{k-1}} - \delta_{i_k, i_{k-1}-1} - \delta_{i_k, i_{k-1}+1} - 2\delta_{i_k, -i_{k-1}} = -h_{k-1}(\mathbf{j}) + 1,$$

when $|i_k + i_{k-1}| = 1$.

In a more concrete form, if \mathbf{i} is a residue sequence satisfies (6.5.2.4) and does not satisfy the assumption of Lemma 7.56, we have $|i_{k-1} - i_k| \neq |i_{k-1} - i_{k+1}|$, and

$$h_k(\mathbf{i}) = \begin{cases} -h_{k-1}(\mathbf{j}) - 1, & \text{if } |i_{k-1} - i_k| = 1, \\ -h_{k-1}(\mathbf{j}) + 1, & \text{if } |i_{k-1} - i_{k+1}| = 1. \end{cases}$$

Following the similar argument, if \mathbf{i} is a residue sequence satisfies (6.5.2.6) and does not satisfy the assumption of Lemma 7.57, we have $|i_k - i_{k-1}| \neq |i_k - i_{k+1}|$, and

$$\begin{cases} h_{k+1}(\mathbf{i}) = h_{k-1}(\mathbf{j}) - 3, & \text{if } |i_k - i_{k+1}| = 1, \\ h_{k+1}(\mathbf{j}) = h_{k-1}(\mathbf{i}) - 3, & \text{if } |i_k - i_{k-1}| = 1. \end{cases}$$

Now we proceed to prove all the other cases.

7.59. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_k + i_{k+1} = 0$ and $|i_{k-1} - i_{k+1}| = 1$. By excluding the assumptions of Lemma 7.56, we have

$$\begin{aligned} e(\mathbf{i})\psi_k\psi_{k-1}\psi_k &= e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = e(\mathbf{i})\epsilon_{k-1}e(\mathbf{i} \cdot s_k s_{k-1} s_k) = 0, \\ \psi_k\psi_{k-1}\psi_k e(\mathbf{i}) &= \psi_{k-1}\psi_k\psi_{k-1}e(\mathbf{i}) = e(\mathbf{i} \cdot s_k s_{k-1} s_k)\epsilon_{k-1}\epsilon_k e(\mathbf{i}) = 0. \end{aligned}$$

Proof. Let $\mathbf{j} = \mathbf{i} \cdot s_k s_{k-1} s_k$. Because the assumptions of Lemma 7.56 are excluded, by Remark 7.58, we have $h_k(\mathbf{i}) = -h_{k-1}(\mathbf{j}) + 1$. Since $\mathbf{i} \in I^n$, by Lemma 3.6, we have $h_k(\mathbf{i}) \leq 0$, which implies $h_{k-1}(\mathbf{j}) \geq 1$. Therefore, by Lemma 3.6, we have $\mathbf{j} \notin I^n$. Then by Lemma 6.1, we have $e(\mathbf{j}) = 0$. The Lemma follows by Proposition 7.11. \square

7.60. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_k + i_{k+1} = 0$ and $|i_{k-1} - i_k| = 1$. By excluding the assumptions of Lemma 7.56, if $\mathbf{i} \cdot s_{k-1} s_k s_{k-1} \in I^n$, then there is exactly one of $\mathbf{i} \cdot s_k$ and $\mathbf{i} \cdot s_{k-1} s_k$ in I^n .

Proof. Let $\mathbf{j} = \mathbf{i} \cdot s_{k-1} s_k s_{k-1}$. Because the assumptions of Lemma 7.56 are excluded, by Remark 7.58, we have $h_k(\mathbf{i}) = -h_{k-1}(\mathbf{j}) - 1$. Moreover, as $\mathbf{i} \cdot s_{k-1} s_k = \mathbf{j} \cdot s_{k-1}$, we have $h_k(\mathbf{i}) + h_k(\mathbf{i} \cdot s_k) = 0$ and $h_{k-1}(\mathbf{j}) + h_{k-1}(\mathbf{i} \cdot s_{k-1} s_k) = 0$.

If $\mathbf{i} \cdot s_k \in I^n$, as $\mathbf{i} \in I^n$, we have $h_k(\mathbf{i}) = h_k(\mathbf{i} \cdot s_k) = 0$. Hence, we have $h_k(\mathbf{j}) = -1$, which implies that $h_k(\mathbf{i} \cdot s_{k-1} s_k) = 1 > 0$. Therefore, $\mathbf{i} \cdot s_{k-1} s_k \notin I^n$.

If $\mathbf{i} \cdot s_k \notin I^n$, by Lemma 3.20, we have $h_k(\mathbf{i}) \neq 0$. By Lemma 3.6, we have $-2 \leq h_k(\mathbf{i}) \leq -1$, which implies $0 \leq h_{k-1}(\mathbf{j}) \leq 1$. As $\mathbf{j} \in I^n$, it forces $h_{k-1}(\mathbf{j}) = 0$ by Lemma 3.6. Hence, by Lemma 3.20, we have $\mathbf{i} \cdot s_{k-1} s_k = \mathbf{j} \cdot s_{k-1} \in I^n$.

Hence, we have $\mathbf{i} \cdot s_k \in I^n$ if and only if $\mathbf{i} \cdot s_{k-1} s_k \notin I^n$, which proves the Lemma. \square

By Lemma 7.60, it is equivalent to say that under the assumptions of Lemma 7.60, we have either $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = 0$ or $e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$; similarly, we have either $\psi_k\psi_{k-1}\psi_k e(\mathbf{i}) = 0$ or $\psi_{k-1}\psi_k\psi_{k-1} e(\mathbf{i}) = 0$.

7.61. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_k + i_{k+1} = 0$ and $|i_{k-1} - i_k| = 1$. By excluding the assumptions of Lemma 7.56, we have

$$e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = \begin{cases} e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} + e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} = i_k - 1, \\ e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} - e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} = i_k + 1. \end{cases}$$

Proof. Define $\mathbf{j} = \mathbf{i}\cdot s_k s_{k-1} s_k$. By Proposition 7.11, we have $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_k\psi_{k-1}\psi_k e(\mathbf{j})$ and $e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} e(\mathbf{j})$. We assume $\mathbf{j} \in I^n$, as otherwise, we have $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{j}) = 0$, and the Lemma follows.

By Lemma 7.60, there is exactly one of $\mathbf{i}\cdot s_k$ and $\mathbf{i}\cdot s_{k-1} s_k$ in I^n . Suppose $\mathbf{i}\cdot s_k \in I^n$. Then we have $e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$. As $\mathbf{i}\cdot s_k \in I^n$, by Lemma 3.20, we have $h_k(\mathbf{i}) = 0$. Because $|i_{k-1} - i_k| = 1$, by Remark 7.58, we have $h_{k-1}(\mathbf{j}) = -h_k(\mathbf{i}) - 1 = -1$. As $j_{k-1} = i_{k+1}$ and the assumptions of Lemma 7.56 are excluded, we have $j_{k-1} \neq 0, \pm \frac{1}{2}$ as $i_k \neq 0, \pm \frac{1}{2}$. Hence, $\mathbf{j} \in I^n_{k-1,0}$. Therefore, by (3.15), Corollary 7.48, Proposition 7.17 and Proposition 7.35, we have

$$\begin{aligned} e(\mathbf{i})\psi_k\psi_{k-1}\psi_k &= (-1)^{a_{k-1}(\mathbf{j})} e(\mathbf{i})\psi_k\psi_{k-1}\psi_k e(\mathbf{j})\epsilon_{k-1}e(\mathbf{j}) = (-1)^{a_{k-1}(\mathbf{j})} e(\mathbf{i})\psi_k\psi_{k-1}^2\epsilon_k e(\mathbf{j})\epsilon_{k-1}e(\mathbf{j}) \\ &= (-1)^{a_{k-1}(\mathbf{j})} e(\mathbf{i})\psi_k\epsilon_k e(\mathbf{j})\epsilon_{k-1}e(\mathbf{j}) = (-1)^{a_{k-1}(\mathbf{j})+a_k(\mathbf{i}\cdot s_k)} e(\mathbf{i})\epsilon_k e(\mathbf{j})\epsilon_{k-1}e(\mathbf{j}). \end{aligned}$$

By direct calculation, we have $(-1)^{a_k(\mathbf{i}\cdot s_k)+a_{k-1}(\mathbf{j})} = 1$ when $i_{k-1} = i_k - 1$ and $(-1)^{a_k(\mathbf{i}\cdot s_k)+a_{k-1}(\mathbf{j})} = -1$ when $i_{k-1} = i_k + 1$. As $e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$, the Lemma follows when $\mathbf{i}\cdot s_k \in I^n$.

Suppose $\mathbf{i}\cdot s_{k-1} s_k \in I^n$. Then we have $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = 0$. As $\mathbf{i}\cdot s_{k-1} s_k = \mathbf{j}\cdot s_{k-1} \in I^n$, by Lemma 3.20, we have $h_{k-1}(\mathbf{j}) = 0$, which implies $h_k(\mathbf{i}) = -1$ by Remark 7.58. Because the assumptions of Lemma 7.56 are excluded, we have $i_k \neq 0, \pm \frac{1}{2}$. Hence $\mathbf{i} \in I^n_{k,0}$. Following the similar argument as before, we have

$$e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = (-1)^{a_k(\mathbf{i})+a_{k-1}(\mathbf{j}\cdot s_{k-1})} e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{j}).$$

By direct calculation, we have $(-1)^{a_k(\mathbf{i}\cdot s_k)+a_{k-1}(\mathbf{j})} = -1$ when $i_{k-1} = i_k - 1$ and $(-1)^{a_k(\mathbf{i}\cdot s_k)+a_{k-1}(\mathbf{j})} = 1$ when $i_{k-1} = i_k + 1$. As $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = 0$, the Lemma follows when $\mathbf{i}\cdot s_{k-1} s_k \in I^n$. \square

The next Lemma follows by almost the same method as Lemma 7.61.

7.62. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_k + i_{k+1} = 0$ and $|i_{k-1} - i_k| = 1$. By excluding the assumptions of Lemma 7.56, we have

$$\psi_k\psi_{k-1}\psi_k e(\mathbf{i}) = \begin{cases} \psi_{k-1}\psi_k\psi_{k-1}e(\mathbf{i}) + e(\mathbf{i}\cdot s_k s_{k-1} s_k)\epsilon_{k-1}\epsilon_k e(\mathbf{i}), & \text{if } i_{k-1} = i_k - 1, \\ \psi_{k-1}\psi_k\psi_{k-1}e(\mathbf{i}) - e(\mathbf{i}\cdot s_k s_{k-1} s_k)\epsilon_{k-1}\epsilon_k e(\mathbf{i}), & \text{if } i_{k-1} = i_k + 1. \end{cases}$$

By Lemma 7.56, Lemma 7.59 and Lemma 7.61, (6.5.2.4) has been proved; and by Lemma 7.56, Lemma 7.59 and Lemma 7.62, (6.5.2.5) has been proved. It left us to prove (6.5.2.6).

7.63. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfying (6.5.2.6). Then we have $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$.

Proof. When \mathbf{i} is under the assumptions of Lemma 7.57, the Lemma holds. Hence we consider the cases excluding the assumptions of Lemma 7.57.

Let $\mathbf{j} = \mathbf{i}\cdot s_k s_{k-1} s_k$. If $\mathbf{j} \notin I^n$, we have $e(\mathbf{j}) = 0$ by Lemma 6.1, and the Lemma follows by Proposition 7.11. Assume $\mathbf{j} \in I^n$ and write

$$\begin{aligned} \mathbf{i} &= (i_1, \dots, i_{k-2}, i_{k-1}, i_k, i_{k+1}, i_{k+2}, \dots, i_n), \\ \mathbf{j} &= (i_1, \dots, i_{k-2}, i_{k+1}, i_k, i_{k-1}, i_{k+2}, \dots, i_n). \end{aligned}$$

As $i_{k-1} + i_{k+1} = 0$, we have $h_{k-1}(\mathbf{i}) = -h_{k-1}(\mathbf{j})$. Because $\mathbf{i}, \mathbf{j} \in I^n$, we have $-2 \leq h_{k-1}(\mathbf{i}), h_{k-1}(\mathbf{j}) \leq 0$ by Lemma 3.6, which forces $h_{k-1}(\mathbf{i}) = h_{k-1}(\mathbf{j}) = 0$. As the assumptions of Lemma 7.57 are excluded, by Remark 7.58, we have either $h_{k+1}(\mathbf{i}) = -3$ or $h_{k+1}(\mathbf{j}) = -3$. Hence, by Lemma 3.6, we have $\mathbf{i} \notin I^n$ or $\mathbf{j} \notin I^n$, which leads to contradiction. Hence, we always have $\mathbf{j} \notin I^n$, and the Lemma follows. \square

Combining Lemma 7.53 - 7.63, we have the following Lemma.

7.64. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfies (6.5.2). Then we have

$$e(\mathbf{i})\mathcal{B}_k = \begin{cases} e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k - 1), & (7.44) \\ -e(\mathbf{i})\epsilon_k\epsilon_{k-1}e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_k + i_{k+1} = 0 \text{ and } i_{k-1} = \pm(i_k + 1), & (7.45) \\ e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k - 1), & (7.46) \\ -e(\mathbf{i})\epsilon_{k-1}\epsilon_k e(\mathbf{i}\cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} + i_k = 0 \text{ and } i_{k+1} = \pm(i_k + 1), & (7.47) \\ 0, & \text{otherwise,} & (7.48) \end{cases}$$

where $\mathcal{B}_k = \psi_k\psi_{k-1}\psi_k - \psi_{k-1}\psi_k\psi_{k-1}$.

Proof. Suppose $i_k + i_{k+1} = 0$, and $i_{k-1} = \pm(i_k - 1)$ or $i_{k-1} = \pm(i_k + 1)$. Then \mathbf{i} satisfy (6.5.2.4). Hence, by Lemma 7.56, Lemma 7.59 and Lemma 7.61, (7.44) and (7.45) hold.

Suppose $i_{k-1} + i_k = 0$, and $i_{k+1} = \pm(i_k - 1)$ or $i_{k+1} = \pm(i_k + 1)$. Then \mathbf{i} satisfy (6.5.2.5). Hence, by Lemma 7.56, Lemma 7.59 and Lemma 7.62, (7.46) and (7.47) hold.

For the rest of the cases, when \mathbf{i} satisfies (6.5.2.1) - (6.5.2.3), by Lemma 7.53 - 7.55, we have

$$e(\mathbf{i})^\mathcal{O} \psi_k^\mathcal{O} \psi_{k-1}^\mathcal{O} \psi_k^\mathcal{O} = e(\mathbf{i})^\mathcal{O} \psi_{k-1}^\mathcal{O} \psi_k^\mathcal{O} \psi_{k-1}^\mathcal{O},$$

which proves $e(\mathbf{i})\mathcal{B}_k = 0$ by lifting the elements into $\mathcal{B}_n(\delta)$; and when \mathbf{i} satisfies (6.5.2.6), by Lemma 7.63, we have $e(\mathbf{i})\mathcal{B}_k = 0$. \square

Finally, we prove (6.5.3). First we consider some special cases.

7.65. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$. If $i_{k-1} = -i_k = i_{k+1} = \pm\frac{1}{2}$, we have $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$.

Proof. By the definition of h_k , as $i_{k-1} = i_{k+1}$, we have $h_k(\mathbf{i} \cdot s_k) = h_{k-1}(\mathbf{i} \cdot s_k) - 3$, which implies $\mathbf{i} \cdot s_k \notin I^n$ by Lemma 3.6. Hence $e(\mathbf{i} \cdot s_k) = 0$ by Lemma 6.1 and $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = 0$ by Proposition 7.11. Similarly, we have $h_{k+1}(\mathbf{i} \cdot s_{k-1}) = h_k(\mathbf{i} \cdot s_{k-1}) - 3$. Following the same process we have $e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$. \square

7.66. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$. If $i_{k-1} = i_k = i_{k+1} = 0$, we have $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$.

Proof. By Lemma 7.13, we have $e(\mathbf{i})\psi_k = e(\mathbf{i})\psi_{k-1} = 0$, which proves the Lemma. \square

7.67. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = i_k = -i_{k+1}$. Then we have

$$e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0,$$

$$\psi_k\psi_{k-1}\psi_k e(\mathbf{i}) = \psi_{k-1}\psi_k\psi_{k-1} e(\mathbf{i}) = 0.$$

Proof. By Lemma 7.66, when $i_{k-1} = i_k = 0$ the Lemma follows. When $i_{k-1} = i_k \neq 0$, we have $h_k(\mathbf{i}) = h_{k-1}(\mathbf{i}) + 2$ if $i_{k-1} \neq \pm\frac{1}{2}$ and $h_k(\mathbf{i}) = h_{k-1}(\mathbf{i}) + 3$ if $i_{k-1} = \pm\frac{1}{2}$. As $\mathbf{i} \in I^n$, by Lemma 3.6, we have $-2 \leq h_{k-1}(\mathbf{i}), h_k(\mathbf{i}) \leq 0$, which forces $h_{k-1}(\mathbf{i}) = -2$.

Let $\mathbf{j} = \mathbf{i} \cdot s_k s_{k-1} s_k$. We have $h_{k-1}(\mathbf{j}) = -h_{k-1}(\mathbf{i}) \geq 2$, which implies $\mathbf{j} \notin I^n$ by Lemma 3.6, and hence, $e(\mathbf{j}) = 0$ by Lemma 6.1. The Lemma holds by Proposition 7.11. \square

Lemma 7.67 proves (6.5.3) when $i_{k-1} = i_k = -i_{k+1}$ and $-i_{k-1} = i_k = i_{k+1}$; and Lemma 7.65 and Lemma 7.66 prove (6.5.3) when $i_{k-1} = -i_k = i_{k+1} = 0$ or $\pm\frac{1}{2}$. It only left us to consider when $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$.

7.68. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$. Then at most one of $\mathbf{i} \cdot s_k$ and $\mathbf{i} \cdot s_{k-1}$ is in I^n .

Proof. Because $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$, we have $h_{k-1}(\mathbf{i}) = h_{k-1}(\mathbf{i} \cdot s_k) = h_k(\mathbf{i} \cdot s_k) - 2$. Assume $\mathbf{i} \cdot s_k \in I^n$. Then by Lemma 3.6, we have $-2 \leq h_{k-1}(\mathbf{i}), h_k(\mathbf{i} \cdot s_k) \leq 0$, which forces $h_{k-1}(\mathbf{i}) = -2$. Hence $h_{k-1}(\mathbf{i} \cdot s_{k-1}) = -h_{k-1}(\mathbf{i}) = 2$. By Lemma 3.6, we have $\mathbf{i} \cdot s_{k-1} \notin I^n$, which completes the proof. \square

So we will consider 3 cases: $\mathbf{i} \cdot s_k \in I^n$, $\mathbf{i} \cdot s_{k-1} \in I^n$, and both $\mathbf{i} \cdot s_k, \mathbf{i} \cdot s_{k-1} \notin I^n$. Note that by Lemma 3.20, we have $\mathbf{i} \cdot s_k \in I^n$ if and only if $h_k(\mathbf{i}) = h_k(\mathbf{i} \cdot s_k) = 0$ and $\mathbf{i} \cdot s_{k-1}$ if and only if $h_{k-1}(\mathbf{i}) = h_{k-1}(\mathbf{i} \cdot s_{k-1}) = 0$.

7.69. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$. If $h_k(\mathbf{i}) = 0$, for $\mathbf{t} \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $t(k-1) = t(k+1)$, we have

$$f_{\mathbf{t}} \psi_k^\mathcal{O} \psi_{k-1}^\mathcal{O} \psi_k^\mathcal{O} e(\mathbf{i})^\mathcal{O} = (-1)^{a_{k-1}(\mathbf{i})+1} f_{\mathbf{t}} e_{k-1}^\mathcal{O} e(\mathbf{i})^\mathcal{O}.$$

Proof. Because $h_k(\mathbf{i}) = 0$ and $i_{k-1} = -i_k$, by Lemma 3.11 we have $t(k) = -t(k-1)$. Hence, as $t(k-1) = t(k+1)$, we have $t(k-1) = -t(k) = t(k+1)$.

Write $\mathbf{t} = (\alpha_1, \dots, \alpha_n)$ and let $\alpha = t(k-1)$. Hence, we have $\alpha = \gamma_1$ or $\alpha = -\gamma_2$. Let β be a general nodes such that $\beta = -\gamma_2$ if $\alpha = \gamma_1$, and $\beta = \gamma_1$ if $\beta = -\gamma_2$. Define an up-down tableau $\mathbf{s} = (\beta_1, \dots, \beta_n)$ such that $\mathbf{s} \stackrel{k-1}{\sim} \mathbf{t}$, and $\beta_{k-1} = \beta, \beta_k = -\beta$. Hence, we can write

$$\mathbf{t} = (\alpha_1, \dots, \alpha_{k-2}, \alpha, -\alpha, \alpha, \alpha_{k+2}, \dots, \alpha_n),$$

$$\mathbf{s} = (\alpha_1, \dots, \alpha_{k-2}, \beta, -\beta, \alpha, \alpha_{k+2}, \dots, \alpha_n).$$

Because γ_1 and γ_2 are not adjacent, i.e. $-\beta$ and α are not adjacent, by Lemma 2.6, $\mathbf{v} = \mathbf{s} \cdot s_k$ is an up-down tableau, and we can write

$$\mathbf{v} = (\alpha_1, \dots, \alpha_{k-2}, \beta, \alpha, -\beta, \alpha_{k+2}, \dots, \alpha_n);$$

and because $-\beta + \alpha \neq 0$, by Lemma 2.6, $\mathbf{u} = \mathbf{v} \cdot s_{k-1}$ is an up-down tableau, and we can write

$$\mathbf{u} = (\alpha_1, \dots, \alpha_{k-2}, \alpha, \beta, -\beta, \alpha_{k+2}, \dots, \alpha_n).$$

Notice that $u \stackrel{k}{\sim} v$ and $u \in \mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$. Because $h_k(\mathbf{i}) = 0$, by Lemma 3.10, u is the unique up-down tableau in $\mathcal{T}_n^{ud}(\mathbf{i} \cdot s_k)$ such that $u \stackrel{k}{\sim} t$. Hence, by Lemma 7.2 and Lemma 7.4, we have

$$f_{tt}\psi_k^\mathcal{O}\psi_{k-1}^\mathcal{O}\psi_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} = f_{tt}\psi_k^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_{k-1}^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_k^\mathcal{O}\frac{f_{tt}}{\gamma_t} + f_{tt}\psi_k^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_{k-1}^\mathcal{O}\frac{f_{vv}}{\gamma_v}\psi_k^\mathcal{O}\frac{f_{ss}}{\gamma_s}. \quad (7.49)$$

First we work with the second term of (7.49). By Lemma 7.4, Lemma 7.42 and Lemma 7.15, we have

$$\begin{aligned} f_{tt}\psi_k^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_{k-1}^\mathcal{O}\frac{f_{vv}}{\gamma_v}\psi_k^\mathcal{O}\frac{f_{ss}}{\gamma_s} &= \frac{1}{c_t(k) + c_u(k)} f_{tt}\epsilon_k^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_{k-1}^\mathcal{O}\frac{f_{vv}}{\gamma_v}\psi_k^\mathcal{O}\frac{f_{ss}}{\gamma_s} \\ &= \frac{1}{c_t(k) + c_u(k)} f_{tt}\epsilon_k^\mathcal{O}\epsilon_{k-1}^\mathcal{O}(\psi_k^\mathcal{O})^2\frac{f_{ss}}{\gamma_s} = \frac{1}{c_t(k) + c_u(k)} f_{tt}\epsilon_k^\mathcal{O}\epsilon_{k-1}^\mathcal{O}\frac{f_{ss}}{\gamma_s}. \end{aligned}$$

Because $i_k \neq 0, \pm\frac{1}{2}$ and $h_k(\mathbf{i}) = 0$, we have $\mathbf{i} \in I_{k,+}^n$. Hence, by Lemma 6.12 and Lemma 7.21, we have

$$\begin{aligned} f_{tt}\psi_k^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_{k-1}^\mathcal{O}\frac{f_{vv}}{\gamma_v}\psi_k^\mathcal{O}\frac{f_{ss}}{\gamma_s} &= \frac{1}{c_t(k) + c_u(k)} f_{tt}\epsilon_k^\mathcal{O}\frac{f_{tt}}{\gamma_t}\epsilon_{k-1}^\mathcal{O}\frac{f_{ss}}{\gamma_s} \\ &= (-1)^{a_k(\mathbf{i})} \frac{2(c_t(k) - i_k)}{c_t(k) + c_u(k)} f_{tt}\epsilon_{k-1}^\mathcal{O}\frac{f_{ss}}{\gamma_s} = (-1)^{a_{k-1}(\mathbf{i})+1} \frac{2(c_t(k) - i_k)}{c_t(k) + c_u(k)} f_{tt}\epsilon_{k-1}^\mathcal{O}\frac{f_{ss}}{\gamma_s}. \end{aligned} \quad (7.50)$$

Then we work with the first term of (7.49). By Lemma 7.4 and Lemma 7.15, we have

$$f_{tt}\psi_k^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_{k-1}^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_k^\mathcal{O}\frac{f_{tt}}{\gamma_t} = \frac{1}{c_u(k) - c_u(k-1)} \frac{f_{tt}}{\gamma_t} (\psi_k^\mathcal{O})^2 \frac{f_{tt}}{\gamma_t} = \frac{1}{c_u(k) - c_u(k-1)} f_{tt}.$$

Because $i_{k-1} \neq 0, \pm\frac{1}{2}$ and $h_{k-1}(\mathbf{i}) = -2$, we have $\mathbf{i} \in I_{k,-}^n$. Hence by Lemma 6.12 and Lemma 7.4, we have

$$f_{tt}\psi_k^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_{k-1}^\mathcal{O}\frac{f_{uu}}{\gamma_u}\psi_k^\mathcal{O}\frac{f_{tt}}{\gamma_t} = \frac{1}{c_u(k) - c_u(k-1)} f_{tt} = (-1)^{a_{k-1}(\mathbf{i})+1} \frac{2(c_t(k-1) - i_{k-1})}{c_u(k-1) - c_u(k)} f_{tt}\epsilon_{k-1}^\mathcal{O}\frac{f_{tt}}{\gamma_t}. \quad (7.51)$$

By the definitions, we have $\alpha > 0$ if $\beta < 0$ and $\alpha < 0$ if $\beta > 0$. Hence, we have $2(c_t(k) - i_k) = c_t(k) + c_u(k)$ and $2(c_t(k-1) - i_{k-1}) = c_u(k-1) - c_u(k)$. Therefore, we have

$$\frac{2(c_t(k) - i_k)}{c_t(k) + c_u(k)} = \frac{2(c_t(k-1) - i_{k-1})}{c_u(k-1) - c_u(k)} = 1.$$

Because $h_{k-1}(\mathbf{i}) = -2$, by Lemma 3.10, s is the unique up-down tableau in $\mathcal{T}_n^{ud}(\mathbf{i})$ such that $s \stackrel{k-1}{\sim} t$ and $s \neq t$. Hence, we have $f_{tt}\epsilon_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} = f_{tt}\epsilon_k^\mathcal{O}\left(\frac{f_{tt}}{\gamma_t} + \frac{f_{ss}}{\gamma_s}\right)$. Substituting (7.50) and (7.51) into (7.49), the Lemma follows \square

Following the similar argument as above, we have the next Lemma.

7.70. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$. If $h_k(\mathbf{i}) = 0$, for $t \in \mathcal{T}_n^{ud}(\mathbf{i})$ with $t(k-1) \neq t(k+1)$, we have

$$f_{tt}\psi_k^\mathcal{O}\psi_{k-1}^\mathcal{O}\psi_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} = (-1)^{a_{k-1}(\mathbf{i})+1} f_{tt}\epsilon_{k-1}^\mathcal{O}e(\mathbf{i})^\mathcal{O}.$$

Combining Lemma 7.69 and Lemma 7.70, we have the next Lemma.

7.71. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$. If $h_k(\mathbf{i}) = 0$, we have

$$e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} - (-1)^{a_{k-1}(\mathbf{i})}e(\mathbf{i})\epsilon_{k-1}e(\mathbf{i}).$$

Proof. Choose arbitrary $t \in \mathcal{T}_n^{ud}(\mathbf{i})$. By Lemma 7.69 and Lemma 7.70, we have

$$f_{tt}\psi_k^\mathcal{O}\psi_{k-1}^\mathcal{O}\psi_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} = (-1)^{a_{k-1}(\mathbf{i})+1} f_{tt}\epsilon_{k-1}^\mathcal{O}e(\mathbf{i})^\mathcal{O},$$

which implies

$$e(\mathbf{i})^\mathcal{O}\psi_k^\mathcal{O}\psi_{k-1}^\mathcal{O}\psi_k^\mathcal{O}e(\mathbf{i})^\mathcal{O} = (-1)^{a_{k-1}(\mathbf{i})+1}e(\mathbf{i})^\mathcal{O}\epsilon_{k-1}^\mathcal{O}e(\mathbf{i})^\mathcal{O}.$$

The Lemma follows by lifting the elements from $\mathcal{B}_n^\mathcal{O}(x)$ to $\mathcal{B}_n(\delta)$. \square

The next Lemma is proved following the similar argument as Lemma 7.71.

7.72. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$. If $h_{k-1}(\mathbf{i}) = 0$, we have

$$e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} + (-1)^{a_k(\mathbf{i})}e(\mathbf{i})\epsilon_k e(\mathbf{i}).$$

The only left case is that when both of $\mathbf{i} \cdot s_k$ and $\mathbf{i} \cdot s_{k-1}$ are not in I^n .

7.73. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ with $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm\frac{1}{2}$. If $h_{k-1}(\mathbf{i}), h_k(\mathbf{i}) \neq 0$, we have

$$e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0.$$

Proof. As $\mathbf{i} \cdot s_k, \mathbf{i} \cdot s_{k-1} \notin I^n$, we have $e(\mathbf{i})\psi_k = e(\mathbf{i})\psi_{k-1} = 0$ by Lemma 6.1 and Proposition 7.11, which implies $e(\mathbf{i})\psi_k\psi_{k-1}\psi_k = e(\mathbf{i})\psi_{k-1}\psi_k\psi_{k-1} = 0$. \square

7.74. Lemma. Suppose $1 < k < n$ and $\mathbf{i} \in I^n$ satisfies (6.5.3). Then we have

$$e(\mathbf{i})\mathcal{B}_k = \begin{cases} -(-1)^{a_{k-1}(\mathbf{i})} e(\mathbf{i})\epsilon_{k-1} e(\mathbf{i} \cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm \frac{1}{2} \text{ and } h_k(\mathbf{i}) = 0, \\ (-1)^{a_{k-1}(\mathbf{i})} e(\mathbf{i})\epsilon_k e(\mathbf{i} \cdot s_k s_{k-1} s_k), & \text{if } i_{k-1} = -i_k = i_{k+1} \neq 0, \pm \frac{1}{2} \text{ and } h_{k-1}(\mathbf{i}) = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{B}_k = \psi_k\psi_{k-1}\psi_k - \psi_{k-1}\psi_k\psi_{k-1}$.

Proof. If $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm \frac{1}{2}$ and $h_k(\mathbf{i}) = 0$, the Lemma holds by Lemma 7.71; and if $i_{k-1} = -i_k = i_{k+1} \neq 0, \pm \frac{1}{2}$ and $h_{k-1}(\mathbf{i}) = 0$, the Lemma holds by Lemma 7.72; and for the rest of the cases, by Lemma 7.65 - 7.67 and Lemma 7.73, the Lemma holds. \square

Therefore, by combining Lemma 7.50, Lemma 7.64 and Lemma 7.74, we have the following Proposition.

7.75. Proposition. In $\mathcal{B}_n(\delta)$, the braid relations hold.

7.6. The graded cellular basis of $\mathcal{B}_n(\delta)$

Now we are ready to prove our main result of this paper.

7.76. Theorem. Suppose R is a field with characteristic 0 and $\delta \in R$. Then $\mathcal{B}_n(\delta) \cong \mathcal{G}_n(\delta)$.

Proof. We can define a map $\mathcal{G}_n(\delta) \rightarrow \mathcal{B}_n(\delta)$ by sending $e(\mathbf{i})$ to $e(\mathbf{i})$, y_k to y_k , ψ_k to ψ_k and ϵ_k to ϵ_k . By Proposition 7.6, Proposition 7.11, P and Proposition 7.75, the map is a homomorphism. By Proposition 6.20, the map is surjective. By Theorem 5.28, we have $\dim \mathcal{G}_n(\delta) \leq (2n-1)!!$. As $\dim \mathcal{B}_n(\delta) = (2n-1)!!$, it implies the map is an isomorphism. Hence we have $\mathcal{B}_n(\delta) \cong \mathcal{G}_n(\delta)$. \square

7.77. Theorem. Suppose R is a field with characteristic 0 and $\delta \in R$. Then $\mathcal{B}_n(\delta)$ is a graded cellular algebra with a graded cellular basis

$$B = \{ \psi_{\mathbf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda) \}.$$

Proof. By Theorem 5.28, the set B spans $\mathcal{G}_n(\delta)$. By Theorem 7.76, we have $\dim \mathcal{G}_n(\delta) = (2n-1)!!$, which makes B to be a basis of $\mathcal{G}_n(\delta)$. The cellularity is proved by Proposition 5.27. The elements $\psi_{\mathbf{st}}$'s are homogeneous by the construction and we have a degree function \deg on up-down tableaux such that $\deg \psi_{\mathbf{st}} = \deg \mathbf{s} + \deg \mathbf{t}$ by Proposition 4.29. Finally, as we have a $*$ -involution on $\mathcal{G}_n(\delta)$ such that $\psi_{\mathbf{st}}^* = \psi_{\mathbf{ts}}$, one can see that B forms a graded cellular basis of $\mathcal{G}_n(\delta)$. Finally as $\mathcal{G}_n(\delta) \cong \mathcal{B}_n(\delta)$ by Theorem 7.76, we complete the proof. \square

The next Corollary is straightforward by Theorem 7.77.

7.78. Corollary. For any $\mathbf{i} \in P^n$ and $e(\mathbf{i}) \in \mathcal{G}_n(\delta)$, we have $e(\mathbf{i}) \neq 0$ if and only if $\mathbf{i} \in I^n$.

Proof. Suppose \mathbf{i} is the residue sequence of an up-down tableau \mathbf{t} . By Theorem 7.77, we have $\psi_{\mathbf{tt}} \neq 0$. Because $\psi_{\mathbf{tt}} = \psi_{\mathbf{tt}}e(\mathbf{i})$, we have $e(\mathbf{i}) \neq 0$.

Suppose $\mathbf{i} \notin I^n$. For any up-down tableau \mathbf{t} , we have $\psi_{\mathbf{st}}e(\mathbf{i}) = 0$. Therefore, we have $\mathcal{G}_n(\delta)e(\mathbf{i}) = 0$, which implies $e(\mathbf{i}) = 0$. \square

Suppose $E_n(\delta)$ is the two-sided ideal of $\mathcal{G}_n(\delta)$ generated by $\epsilon_1, \dots, \epsilon_{n-1}$ and $E'_n(\delta)$ is the two-sided ideal of $\mathcal{B}_n(\delta)$ generated by e_1, \dots, e_{n-1} . By Remark 6.19, we have $e(\mathbf{i})\epsilon_k e(\mathbf{j}) = e(\mathbf{i})P_k(\mathbf{i})^{-1}e_k Q_k(\mathbf{j})^{-1}e(\mathbf{j})$, which implies that $\epsilon_k \in E'(\delta)$ for any $1 \leq k \leq n-1$. Similarly, we have $e(\mathbf{i})e_k e(\mathbf{j}) = e(\mathbf{i})P_k(\mathbf{i})\epsilon_k Q_k(\mathbf{j})e(\mathbf{j})$, which implies $e_k \in E(\delta)$ for any $1 \leq k \leq n-1$. Therefore, we have $E_n(\delta) \cong E'_n(\delta)$, with isomorphism compatible with $\mathcal{G}_n(\delta) \cong \mathcal{B}_n(\delta)$. Hence, we have $\mathcal{G}_n(\delta)/E_n(\delta) \cong \mathcal{B}_n(\delta)/E'(\delta)$.

Because $E'_n(\delta)$ is the two-sided ideal of $\mathcal{B}_n(\delta)$ generated by e_1, \dots, e_{n-1} , by the definition of $\mathcal{B}_n(\delta)$, we have $\mathcal{B}_n(\delta)/E'(\delta) \cong R \mathfrak{S}_n$. Because R is a field with characteristic 0, by Theorem 2.22, we have $R \mathfrak{S}_n \cong \mathcal{R}_n^\Lambda(R)$ with $\Lambda = \Lambda_k$, for any $k \in \mathbb{Z}$. The next Theorem connects the cyclotomic Khovanov-Lauda-Rouquier algebra and $\mathcal{G}_n(\delta)$.

7.79. Theorem. Suppose $\Lambda = \Lambda_k$ for some $k \in \mathbb{Z}$ and R is a field with characteristic 0. Then we have $\mathcal{G}_n(\delta)/E_n(\delta) \cong \mathcal{R}_n^\Lambda(R)$.

Finally, we remark that for Brauer algebras $\mathcal{B}_n(\delta)$ over fields R with characteristic $p > 0$, or more precisely, for cyclotomic Nazarov-Wenzl algebras $\mathcal{W}_{r,n}(\mathbf{u})$ over arbitrary field R , we should be able to extend the idea of this paper and construct a \mathbb{Z} -graded algebra similar to $\mathcal{G}_n(\delta)$ isomorphic to $\mathcal{W}_{r,n}(\mathbf{u})$. The algebras are generated with elements

$$G_n(\delta) = \{ e(\mathbf{i}) \mid \mathbf{i} \in P^n \} \cup \{ y_k \mid 1 \leq k \leq n \} \cup \{ \psi_k \mid 1 \leq k \leq n-1 \} \cup \{ \epsilon_k \mid 1 \leq k \leq n-1 \},$$

with degrees similar to $\mathcal{G}_n(\delta)$. We are also able to construct a set of homogeneous elements

$$\{ \psi_{\mathbf{st}} \mid (\lambda, f) \in \widehat{B}_n, \mathbf{s}, \mathbf{t} \in \mathcal{T}_n^{ud}(\lambda) \},$$

which forms a graded cellular basis of $\mathcal{W}_{r,n}(\mathbf{u})$.

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